Reverse Mathematics of the Mountain Pass Theorem

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- WKL_0: adds a weak form of König's lemma to RCA_0, stating that every infinite binary tree has an infinite path.

This work is motivated by the use of Reverse Mathematics as a research program within Proof Theory and Computability to investigate the Foundations of *ordinary mathematics*. We recall some ideas on weak subsystems of Second Order Arithmetic:

- RCA_0: Our base system, includes Robinson arithmetic (first-order arithmetic with basic axioms), induction for $\Sigma 1^0$ formulas and the Δ^0_1 comprehension axiom scheme which states the existence of computable sets of natural numbers.
- WKL_0: adds a weak form of König's lemma to RCA_0, stating that every infinite binary tree has an infinite path.
- ACA_0: adds an axiom scheme that allows the construction of sets defined by arithmetical formulas, the arithmetical comprehension axiom scheme.

The Mountain Pass Theorem: Some Previous Definitions

Definition ($C^{1,1}(\mathcal{H}, \mathbb{R})$ functionals)

Let $\mathcal{H}$ be a Hilbert space and $I : \mathcal{H} \to \mathbb{R}$ be a continuous functional. We write $I \in C^1(\mathcal{H}, \mathbb{R})$ if I has a derivative at every point $x \in \mathcal{H}$ and $I' : \mathcal{H} \to \mathcal{H}$ is a continuous function. If in addition, $I' : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz, then we write $I \in C^{1,1}(\mathcal{H}, \mathbb{R})$.

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Definition (Palais-Smale sequence and (PS) condition)

Let $\mathcal{H}$ be a Hilbert space and let $I \in C^1(\mathcal{H}, \mathbb{R})$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$ is *Palais-Smale* if

i. $(I(x_n))_{n\in\mathbb{N}}$ is a bounded sequence, and
ii. $(I'(x_n))_{n\in\mathbb{N}}$ converges to 0.

We say that I satisfies the *Palais-Smale condition* (PS) if every Palais-Smale sequence of elements of $\mathcal{H}$ has a convergent subsequence.

The Mountain Pass Theorem: The Statement

Theorem (MPT, Ambrosetti and Rabinowitz [1], 1973)

Let $\mathcal{H}$ be a Hilbert space and let $I \in C^{1,1}(\mathcal{H}, \mathbb{R})$ be a functional that satisfies (PS). Suppose that $I(0) = 0$ and that the following so-called geometrical hypotheses (GH) are satisfied:
There exist $\rho, \alpha > 0$ such that:

1. *If $\|u\| = \rho$ then $I(u) \geq \alpha$, and*
2. *There is $v \in \mathcal{H}$ such that $\|v\| > \rho$ and $I(v) \leq 0$.*

Then I has a critical value $c_* \geq \alpha$. Moreover, c_* can be characterized as:

$$c_* = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u) = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)),$$

where

$$\Gamma = \{g \in C([0,1], \mathcal{H}) : g(0) = 0,\ g(1) = v\}.$$

The Mountain Pass Theorem: Seeing the Pass

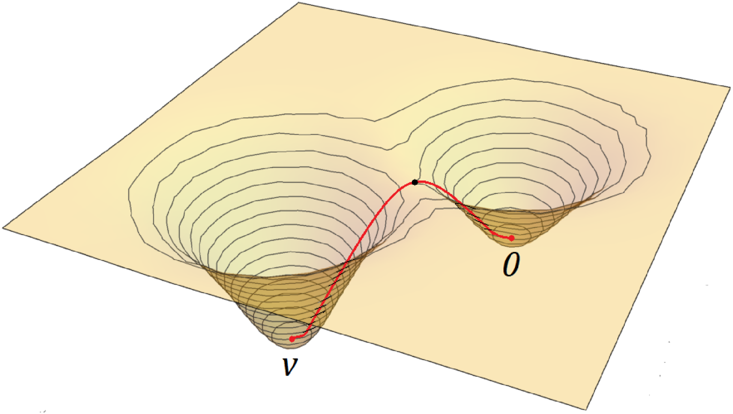

Figure: Graph of a function that satisfies the MPT hypotheses. In the highest point of the red path between the points $I(0) = 0$ and $I(v) < 0$ we can find a mountain pass $c_* = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t))$, where

$\Gamma = \{g \in C([0,1], \mathcal{H}) : g(0) = 0, \; g(1) = v\}$.

Analysis on Subsystems of $\mathbb{Z}_2$

Definition (the system of real numbers)

The following definition is made in RCA_0. A *real number* is defined to be a quickly converging Cauchy sequence of rational numbers $(q_n)_{n\in\mathbb{N}}$, i.e., $\forall n \forall m\,(n > m \rightarrow |q_n - q_m| \leq 2^{-m})$.

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Definition (complete separable metric space)

The following definition is made in RCA_0. A (code for a) *complete separable metric space* $\mathcal{X} = \widehat{X}$ is a non-empty set $X \subseteq \mathbb{N}$ together with a sequence of real numbers $d : X \times X \to \mathbb{R}_{\geq 0}$ such that $d(a, a) = 0$, $d(a, b) = d(b, a)$ and $d(a, c) \leq d(a, b) + d(b, c)$ for all $a, b, c \in X$. A point of $\mathcal{X}$ is a sequence $x = (x_n)_{n\in\mathbb{N}}$ of elements of X such that for all $n > m$, $d(x_n, x_m) \leq 2^{-m}$. We write $x \in \mathcal{X}$ to mean that x is a point of $\mathcal{X}$. We identify $a \in X$ with the constant sequence $(a)_{n\in\mathbb{N}}$ and consider X as a dense subset of $\mathcal{X}$. We set $d(x, y) = \lim_{n\to+\infty} d(x_n, y_n)$, and write $x =_{\mathcal{X}} y$ if $d(x, y) = 0$.

Separable Banach and Hilbert space

Definition (separable Banach space)

The following definition is made in RCA_0. A (code for a) *separable Banach space* $\mathcal{E} = \widehat{E}$ consist in a countable vector space E over the rational field $\mathbb{Q}$ together with a sequence of real numbers $\|\cdot\| : E \to \mathbb{R}$ satisfying $\|q \cdot a\| = |q| \cdot \|a\|$ for all $q \in \mathbb{Q}$ and $a \in E$ and $\|a + b\| \leq \|a\| + \|b\|$ for all $a, b \in E$. As usual, we define a pseudometric on E by $d(a, b) = \|a - b\|$, for all $a, b \in E$. Thus, $\mathcal{E}$ is the complete separable metric space which is the completion of E under d.

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Definition (separable Hilbert space)

The following definition is made in RCA_0. A (code for a real) separable *Hilbert space* $\widehat{H}$ consists of a countable vector space H over $\mathbb{Q}$ together with a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ satisfying for all $x, y, z \in H$ and $a, b \in \mathbb{Q}$: $\langle x, x \rangle \geq 0$, $\langle x, y \rangle = \langle y, x \rangle$, $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$. The space can be viewed as a Banach space with norm $\|x\| = \langle x, x \rangle^{1/2}$.

Towards the Mountain Pass: accessing Γ

Within WKL_0 we can have access to (the code of) the set

$$\Gamma = \widehat{A} = \{g \in C([0,1],\mathcal{H}) : g(0) = 0,\ g(1) = v\}.$$

We can show that it is a separable metric space that is isomorphic to the space of uniformly continuous functions $g : [0,1] \to \mathcal{H}$ such that $g(0) = 0$ and $g(1) = v$; and that the dense set A is given by piece-wise linear continuous functions $p : [0,1] \to \mathcal{H}$ with rational breakpoints, each represented by finitely many pairs $\langle x, p(x)\rangle \in \mathbb{Q} \times H$ and such that $p(0) = 0$ and $p(1) = v$. The metric being defined by $d(f,g) = \max_{t\in[0,1]} \|f(t) - g(t)\|$, which exists due to the following result.

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Theorem (Extreme Value Theorem, Simpson [2])

The following is provable in WKL_0. Let $F : [a,b] \to \mathbb{R}$ be a continuous function. Then F attains a maximum value.

Towards the Mountain Pass Theorem: The Deformation Lemma

Definition (sublevels of a functional)

Let $\mathcal{H}$ be a Hilbert space and let $I : \mathcal{H} \to \mathbb{R}$ be a functional. For every $a \in \mathbb{R}$ we define the *sublevel* of I at a as the following set:

$$I^a = \{x \in \mathcal{H} : I(x) \leq a\}.$$

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Theorem (Deformation Lemma)

The following is provable in RCA_0. Let $\mathcal{H} = \widehat{H}$ be a separable Hilbert space and let $I \in C^{1,1}(\mathcal{H})$ be a functional that satisfies (PS). Suppose $c \in \mathbb{R}$ and $\bar{\varepsilon} > 0$. If c is not a critical value of I, then there exist $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times \mathcal{H}, \mathcal{H})$ such that:

(a) $\eta(1, I^{c+\varepsilon}) \subseteq I^{c-\varepsilon}$,

(b) $\eta(1, u) = u$ *if* $I(u) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$.

Towards the Mountain Pass Theorem: First Obstacles

In order to use the Deformation Lemma to prove the MPT as in the classical proof of the theorem, we should be able to reason to the existence of $c_* = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t))$. Within WKL_0 it is possible to define the following function:

$$\begin{aligned} c : \Gamma &\to \mathbb{R}. \\ g &\mapsto c(g) = \max_{t \in [0,1]} I(g(t)). \end{aligned}$$

However, to find its infimum we would need the power of ACA_0 because of the following:

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Theorem (Sequential Greatest Lower Bound Property, Simpson [2])

The following are equivalent over RCA_0:

1. *ACA_0.*
2. *Every sequence of real numbers bounded from below has an infimum.*

The Mountain Pass Theorem in WKL_0

To remain within WKL_0 we should not treat c_* as an existing real but look for other tools that allow us to prove the existence of a real that once found happens to have the features of an infimum, i.e., it is the greatest lower bound. We begin with the next result.

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Theorem

The following is provable in WKL_0. Let $\mathcal{H} = \widehat{H}$ be a separable Hilbert space and let $I \in C^{1,1}(\mathcal{H}, \mathbb{R})$. Suppose that $I(0) = 0$ and that the GH are satisfied. Consider the notation $c(g) = \max_{t \in [0,1]} I(g(t))$, for $g \in \Gamma$. Then, for all $\varepsilon > 0$ and for all $p \in \Gamma$ there exist $q \in \Gamma$ and $u \in \mathcal{H}$ such that:

i. $\|I'(u)\| < \varepsilon$,

ii. $|c(q) - I(u)| < \varepsilon$, *and*

iii. $c(q) \leq c(p)$.

The Mountain Pass Theorem in WKL_0

Theorem (MPT)

The following is provable in WKL_0. Let $\mathcal{H} = \widehat{H}$ be a separable Hilbert space and let $I \in C^{1,1}(\mathcal{H}, \mathbb{R})$ be a functional that satisfies (PS). Suppose that $I(0) = 0$ and that the GH are satisfied. Then there exists $x_* \in \mathcal{H}$ such that $I'(x_*) = 0$ and

1. *$I(x_*) \leq c(p)$ for all $p \in \Gamma$, and*
2. *if $\beta \in \mathbb{R}$ is such that $\beta \leq c(p)$ for all $p \in \Gamma$, then $I(x_*) \geq \beta$.*

In other words, $I(x_*) = c_*$.

The Mountain Pass Theorem in WKL_0

Sketch of the proof.

Let $\{p_0, p_1, p_2, \dots\}$ be an enumeration of A, the countable dense subset of Γ. Define the following sequence $(q_n)_{n\in\mathbb{N}}$ taking for each $n \in \mathbb{N}$:

$$q_n = p_i, \text{ if } c(p_i) = \min_{j<n} c(p_j).$$

This way, $(c(q_n))_{n\in\mathbb{N}}$ is a decreasing sequence of reals. Let $n \in \mathbb{N}$. Since $2^{-n} > 0$ and $q_n \in \Gamma$, by the previous theorem, there exists $\tilde{q}_n \in \Gamma$ and $x_n \in \mathcal{H}$ such that

i. $\|I'(x_n)\| < 2^{-n}$,

ii. $|c(\tilde{q}_n) - I(x_n)| < 2^{-n}$,

iii. $c(\tilde{q}_n) \le c(q_n)$.

We can prove that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence and thus get a point $x_* \in \mathcal{H}$ such that $I'(x_*) = 0$. Simple but beautiful arguments from Real Analysis get us that $I(x_*) = c_*$. □

The reversal

To prove that $\mathsf{RCA}_0 \vdash \mathsf{MPT} \to \mathsf{WKL}_0$, we proceed by contraposition and show that $\mathsf{RCA}_0 \vdash \neg\mathsf{WKL}_0 \to \neg\mathsf{MPT}$.

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To prove that $\mathsf{RCA}_0 \vdash \mathsf{MPT} \to \mathsf{WKL}_0$, we proceed by contraposition and show that $\mathsf{RCA}_0 \vdash \neg\mathsf{WKL}_0 \to \neg\mathsf{MPT}$. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite binary tree with no branch. Denote by $\widetilde{T}$ the set of *fallen leaves* of T, i.e. $s \in \widetilde{T}$ iff $s \notin T \wedge \forall t\, (t \subsetneq s \to t \in T)$.

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$$a_s = \sum_{i<\mathrm{lh}(s)} \frac{2s(i)}{3^{i+1}} \text{ and } b_s = a_s + \frac{1}{3^{\mathrm{lh}(s)}}.$$

By finding vertices of T level by level, we can enumerate its fallen leaves (binary sequences $u \in \widetilde{T}$) level by level, together with the intervals $[a_u, b_u]$ (a cover of the Cantor set).

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By finding vertices of T level by level, we can enumerate its fallen leaves (binary sequences $u \in \widetilde{T}$) level by level, together with the intervals $[a_u, b_u]$ (a cover of the Cantor set). Thus, in $[0,1]$, we define our function I as follows:

$$I(x) = \mathrm{lh}(u) \text{ if } x = \frac{a_u + b_u}{2} \quad (u \in \widetilde{T}),$$

extending the definition of I at stage $n-1$ on each new interval that appears so to make I continuously differentiable joining those points by adequate pieces of sine functions: $I(x) = a\sin(p(x-h)) + c$.

The reversal: A picture is worth a thousand words

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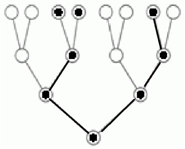
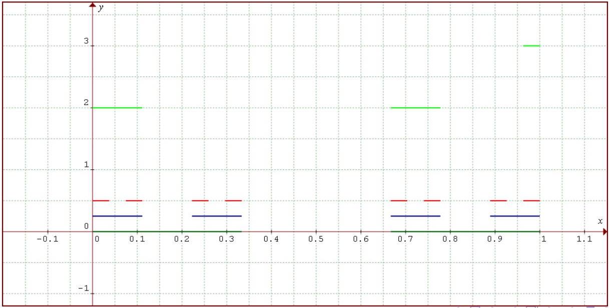
Consider the stage 3 of the construction of the function with the following tree as example:





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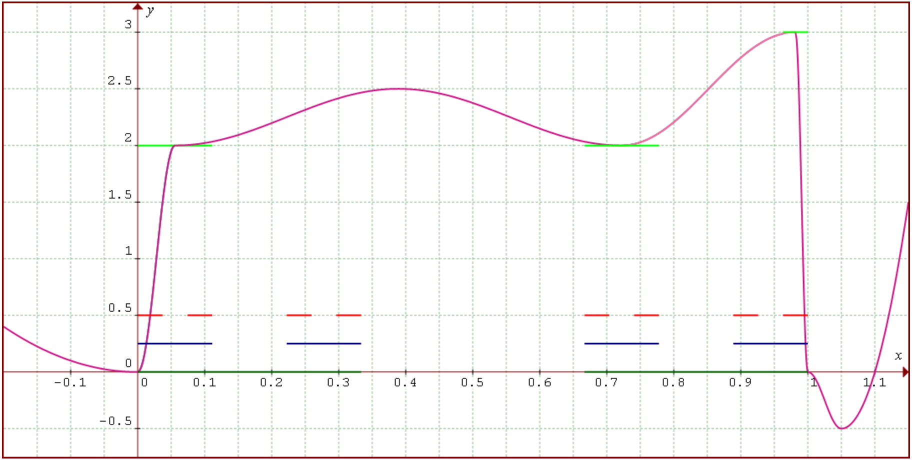

Figure: Stage 3 of the construction of the function I, where we can see it is $C^1(\mathbb{R}, \mathbb{R})$ and the GH of the MPT hold.

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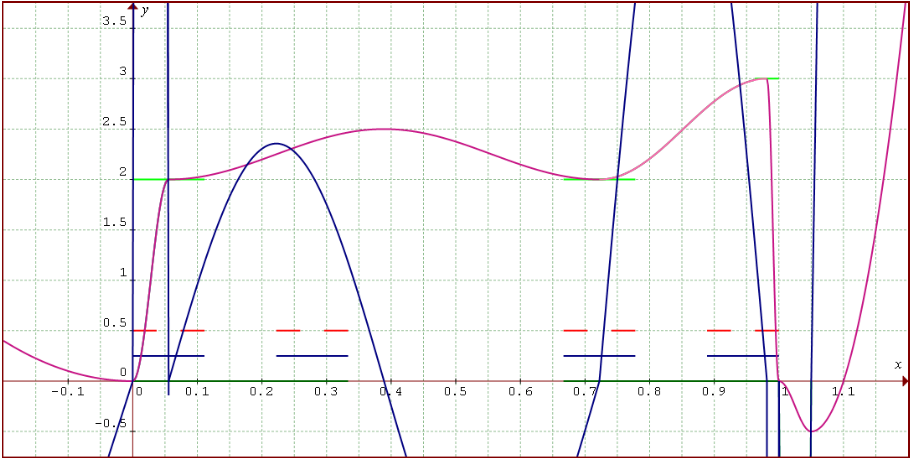

Figure: Stage 3 of the construction of the function I and I', where we can see that $I \in C^{1,1}(\mathbb{R}, \mathbb{R})$ (I' is locally Lipschitz) and also that it satisfies (PS).

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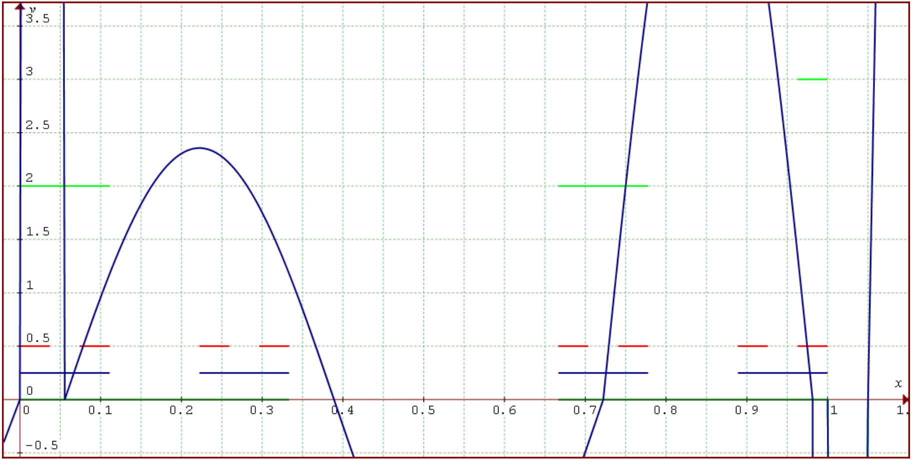

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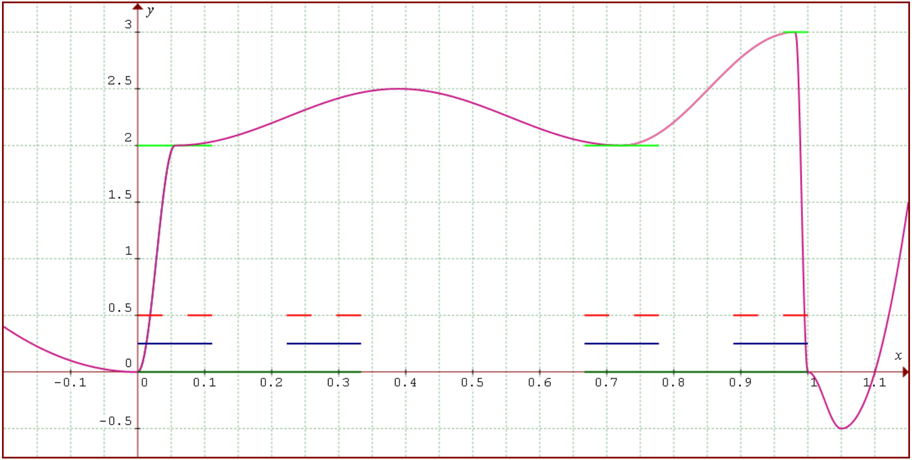

Figure: I satisfies all the hypothesis of the MPT but not its thesis: the continuous paths $p : [0,1] \to \mathbb{R}$ over the graph of I and joining 0 and $I(v)$ are just $I \restriction [0, v]$; but I is unbounded there so $\max_{t\in[0,1]} I(p(t))$ does not exists so no point can have it as image.

Thank you very much!

Bibliography I

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