Algorithmic randomness in harmonic analysis

Diego A. Rojas

(Joint work with Johanna N. Y. Franklin and Lucas E. Rodriguez)

Sam Houston State University

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Part I: Background

Paradigms of Randomness

Algorithmic randomness: The study of random individual elements in sample spaces which pass all effectively devised tests for randomness

Three Randomness Paradigms

- *Incompressibility* (Cannot feasibly compress a random sequence)
- *Unpredictability* (Cannot win against a random sequence in a fair betting game when using a feasible betting strategy)
- *Measure-theoretic typicality* (Random sequences pass all feasible statistical tests)

(X, d, α): Computable metric space

- (X, d): Separable metric space (often complete)
- $\alpha :\subseteq \mathbb{N} \to X$ is an enumeration of a countable dense subset of X
- $d(\alpha(i), \alpha(j))$ is computable uniformly in i and j

We call each $\alpha(i)$ a *rational point* of X.

$(X, \|\cdot\|, \alpha)$: Computably presented Banach space

- X: Separable Banach space over $\mathbb{R}$ or $\mathbb{C}$
- $\alpha :\subseteq \mathbb{N} \to X$ is an enumeration of a countable subset of X whose rational linear span is dense in X.
- For $c_0, \ldots, c_n \in \mathbb{Q}$, $\| \sum_{i=0}^{n} c_i \alpha(j_i) \|$ is computable uniformly in $\langle n, j_0, \ldots, j_n, c_0, \ldots, c_n \rangle$

We call each $\alpha(i)$ a *distinguished vector* of X, and each rational linear combination of them is a *rational vector* of X.

L^p spaces may be viewed both as computable metric (Polish) spaces and as computably presented Banach spaces for computable $p > 1$.

As a computable metric space: $f \in L^p$ is *L^p-computable* if there is a computable sequence $\{f_n\}_n$ of rational **points** in L^p such that $\|f - f_n\|_p < 2^{-n}$ for all $n \in \mathbb{N}$; f is *weakly L^p-computable* if, instead, $\lim_n f_n(x) = f(x)$ pointwise a.e. with $\sum_n \|f_n - f_{n+1}\|_p < \infty$.

As a computably presented Banach space over $\mathbb{R}$ or $\mathbb{C}$: $f \in L^p$ is a *computable vector in L^p* if there is a computable sequence $\{f_n\}_n$ of rational **vectors** in L^p such that $\|f - f_n\|_p < 2^{-n}$ for all $n \in \mathbb{N}$; f is a *weakly computable vector in L^p* if, instead, $\lim_n f_n(x) = f(x)$ pointwise a.e. with $\sum_n \|f_n - f_{n+1}\|_p < \infty$.

Algorithmic Randomness

μ: Computable (Borel) probability measure on X

- $\{q \in \mathbb{Q} : \mu(U) > q\}$ is computably enumerable uniformly in (an index of) an effectively open subset U of X

A μ-*Martin-Löf test* is a sequence $\{U_n\}_{n\in\mathbb{N}}$ of uniformly effectively open subsets of X such that $\mu(U_n) \leq 2^{-n}$ for all $n \in \mathbb{N}$.

A μ-*Schnorr test* is a μ-Martin-Löf test $\{U_n\}_{n\in\mathbb{N}}$ such that $\mu(U_n)$ is computable uniformly in n.

A point $x \in X$ is *Martin-Löf* (*Schnorr*) *random* if $x \notin \bigcap_n U_n$ for any Martin-Löf (Schnorr) test $\{U_n\}_{n\in\mathbb{N}}$.

Part II: Harmonic Analysis and Algorithmic Randomness

“Almost Everywhere” = “Random”

Almost-everywhere theorems in analysis can be used to study algorithmic randomness notions.

Historical Overview

- Demuth 1975: Martin-Löf Randomness and Differentiability of Functions of Bounded Variation
- Kučera 1985: Martin-Löf Randomness and Poincaré Recurrence
- V'yugin 1998: Martin-Löf Randomness and Birkhoff's Ergodic Theorem
- Pathak 2006: Martin-Löf Randomness and Lebesgue Differentiation Theorem
- Brattka-Miller-Nies 2011: Computable/Weak-2 Randomness and Differentiability

Historical Overview (cont.)

- Gács-Hoyrup-Rojas 2011: Schnorr Randomness and Birkhoff's Ergodic Theorem
- Bienvenu-Hölzl-Miller-Nies 2013: Computable Randomness and Denjoy-Young-Saks Theorem
- Pathak-Rojas-Simpson 2014: Schnorr Randomness and Lebesgue Differentiation Theorem
- Miyabe-Nies-Zhang 2016: BSL survey on using a.e. theorems to study randomness
- Franklin-McNicholl-Rute 2016: Schnorr Randomness and Carleson's Theorem in Fourier Analysis
- Many more to come!

Carleson's Theorem in Fourier Analysis

Carleson's Theorem states that the Fourier series of a function $f \in L^p[-\pi, \pi]$, $1 < p < \infty$, converges at almost every $x \in [-\pi, \pi]$. Here, we view $L^p[-\pi, \pi]$ as a computably presented Banach space over $\mathbb{C}$ with rational trigonometric polynomials as the rational vectors.

Theorem. (Franklin, McNicholl, and Rute 2016)

$x \in [-\pi, \pi]$ is Schnorr random if and only if, for any computable vector f in $L^p[-\pi, \pi]$ with $p > 1$ computable, the Fourier series of f converges at x.

- In fact, they proved a stronger converse: If $x \in [-\pi, \pi]$ is not Schnorr random, then there is a computable *function* f such that the Fourier series of f diverges at x.

Carleson's Theorem in Fourier Analysis

Theorem. (Franklin and R. 2025+)

$x \in [-\pi, \pi]$ is Martin-Löf random if and only if, for any weakly computable vector f in $L^p[-\pi, \pi]$ with $p > 1$ computable, the Fourier series of f converges at x.

Proof Sketch.

($\Rightarrow$): If f is a weakly computable vector in L^p with effective approximation $\{\tau_N\}_{N\in\mathbb{N}}$, then $T = \sum_N |\tau_N - \tau_{N+1}|$ is a Martin-Löf integral test. If the Fourier partial sums of f diverge at t_0, then $\{\tau_N\}_{N\in\mathbb{N}}$ diverges at t_0.

($\Leftarrow$): Adapt the construction in FMR16 for Schnorr non-random case, taking into account that we can effectively approximate the measures of the test components without being able to compute them directly. $\square$

The Dirichlet Problem for the Upper Half-Plane

The **Dirichlet Problem for the Upper Half-Plane** goes as follows:

> Given a function f defined everywhere on $\mathbb{R}$, is there a unique continuous function u twice continuously differentiable in $\mathrm{UHP} := \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and continuous on $\partial\,\mathrm{UHP}$, such that u is harmonic in UHP and $u = f$ on $\partial\,\mathrm{UHP}$?

If $f \in L^1(\mathbb{R})$ and, for each $(x, y) \in \mathrm{UHP}$,

$$P[f](x, y) := \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} f(t)\; dt$$

then $\lim_{y\to 0^+} P[f](x, y) = f(x)$ for almost every $x \in \mathbb{R}$. We call $P[f]$ the *Poisson integral* of f.

The Dirichlet Problem for the Upper Half-Plane

We now view $L^1(\mathbb{R})$ as a computable Polish space with compactly-supported piecewise linear functions with rational vertices as the rational points.

Theorem. (Rodriguez and R. 2025+)

If $x \in \mathbb{R}$ is Schnorr random and $f \in L^1(\mathbb{R})$ is L^1-computable then $\lim_{y \to 0^+} P[f](x, y) = f(x)$.

Key Ideas

- To prove this theorem, we need to find a convenient Schnorr test that has nice properties. Since a Schnorr random point x avoids all Schnorr tests, we have creative liberty to find the Schnorr test that we want.
- Once we have this Schnorr test, we take advantage of the L^1-computability of f, specifically its approximating sequence.

The Dirichlet Problem for the Upper Half-Plane

Lemma 1. (Pathak, Rojas, and Simpson 2014)

Let $f \in L^1(\mathbb{R})$ be L^1-computable. Let $\{f_n\}_{n\in\mathbb{N}}$ be a computable name of f. Then we can find uniformly Σ^0_1 sets $\{V_k\}_{k\in\mathbb{N}}$ such that the following statements hold:

1. $\lambda(V_k) \leq \frac{2+\sqrt{2}}{2^{k-1}}$
2. The sequence $\lambda(V_k)$ is uniformly computable
3. $\forall x \notin V_k$ and $n \geq k$ we have

$$|f_i(x) - f_{2n}(x)| \leq \frac{2+\sqrt{2}}{2^n}$$

   For all $i \geq 2n$

Lemma 2. (Rodriguez and R. 2025+)

Let $f \in L^1(\mathbb{R})$ be L^1-computable. Let $\{f_n\}_{n\in\mathbb{N}}$ be a computable name of f. Then we can find uniformly Σ_1^0 sets $\{U_k\}_{k\in\mathbb{N}}$ such that the following statements hold:

1. $\lambda(U_k) \leq \frac{3(\sqrt{2}+2)}{2^k}$
2. The sequence $\lambda(U_k)$ is uniformly computable
3. $\forall x \notin U_k$ and $n \geq k$ we have

$$\int_{\mathbb{R}} P_y(x-t)|f(t) - f_{2n}(t)|\, dt \leq \frac{2+\sqrt{2}}{2^n}$$

Lemma 3. (Rodriguez and R. 2025+)

Let $W_k = U_k \cup V_k$ where U_k is the Schnorr test from Lemma 2 and V_k is the Schnorr test from Lemma 1. If $x \notin \bigcap_{i=0}^{\infty} W_k$, then

$$\lim_{n\to\infty} f_n(x) = f(x) = \lim_{y\to 0^+} P[f](x, y)$$

Theorem. (Rodriguez and R. 2025+)

If $x \in \mathbb{R}$ is not Schnorr random, then there exists an L^1-computable function $f \in L^1(\mathbb{R})$ such that $\lim_{y \to 0^+} P[f](x, y) \neq f(x)$.

Key Ideas

- The idea is to design an L^1-computable sequence of functions that converges effectively to an L^1-computable function f in the limit.
- This function will have the property that the radial limits of the Poisson integral will blow up on Schnorr non-random points x while $f(x)$ is finite.

The Dirichlet Problem for the Upper Half-Plane

Theorem. (Rodriguez and R. 2025+)

If $x \in \mathbb{R}$ is Martin-Löf and $f \in L^1(\mathbb{R})$ is weakly L^1-computable, then $\lim_{y \to 0^+} P[f](x, y) = f(x)$.

Proof Sketch.

Three ingredients: (1) Poisson integral of compactly-supported continuous boundary data fully recovers the data; (2) Poisson integrals of $f - f_n$ converge pointwise to 0; (3) $f(x) = \lim_n f_n(x)$ for every Martin-Löf random $x \in \mathbb{R}$. $\square$

Theorem. (Rodriguez and R. 2025+)

If $x \in \mathbb{R}$ is not Martin-Löf random, then there exists a weakly L^1-computable $f \in L^1(\mathbb{R})$ such that $\lim_{y \to 0^+} P[f](x, y) \neq f(x)$.

Proof Sketch.

Fix $x \in \mathbb{R}$ not Martin-Löf random. Then, there is a universal Martin-Löf test $\{U_k\}_{k\in\mathbb{N}}$ such that $x \in \bigcap_k U_k$. Without loss or generality, we may assume $U_{k+1} \subseteq U_k$ for each k. Since $\{U_k\}_{k\in\mathbb{N}}$ is Σ^0_1 uniformly in k, it is possible to compute an array $\{I_{k,n}\}_{k,n\in\mathbb{N}}$ of rational closed intervals such that $U_k = \bigcup_n I_{k,n}$ for each n and $I_{k,n} \cap I_{k,n'} = \emptyset$ whenever $n \neq n'$. Let $f = \sum_{n,k} 2^{-k}(1 - \mathbf{1}_{I_{k,n}})$. Show that f is weakly $L^1(\mathbb{R})$-computable.

Then, show that $f(x) = 0$ while $\lim_{y\to 0^+} P[f](x, y) > \dfrac{1}{2}$. $\square$

Future Directions

- Look at computable randomness and weak-2 randomness in terms of Carleson's theorem and the Dirichlet problem for UHP
- Study randomness by looking at weak solutions to PDEs, Sobolev spaces, and harmonic analysis literature
- Toward a meta-theorem concerning a.e. theorems and notions of randomness

Thank you!