Type zero choice principle and Halin's infinite ray theorem

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CCR 2025

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1 Infinite Ray Theorem

Definition 1. *The principle* IRT *states: If a graph G has arbitrarily many disjoint rays, then G has infinitely many disjoint rays.*

Definition 2 (IRT)**.** *If a graph G has for all $n \in \mathbb{N}$ a sequence $\langle X_0, \ldots, X_{n-1}\rangle$ of disjoint rays, then G has an infinite sequence $\langle X_0, X_1, \ldots\rangle$ of disjoint rays.*

Definition 3 (IRT^-)**.** *If a graph G has for all $n \in \mathbb{N}$ a sequence $\langle X_0, \ldots, X_{n-1}\rangle$ of disjoint rays, then G has an infinite sequence $\langle G_n\rangle_n$ of disjoint subgraphs each of which contains a ray.*

Definition 4 (Barnes, Goh, Shore [1])**.** *A sentence (theory) T is a theorem (theory) of hyperarithmetic analysis (THA) if*

1. *For every $X \subseteq \mathbb{N}$ $\langle \mathbb{N}, \mathrm{HYP}(X)\rangle \models T$ and*
2. *For every $S \subseteq 2^{\mathbb{N}}$, if $\langle \mathbb{N}, S\rangle \models T$ and $X \in S$ then $\mathrm{HYP}(X) \subseteq S$.*

Definition 5 (ABW)**.** *Given an arithmetic predicate $P(X)$ on $2^{\mathbb{N}}$, either there exists a finite sequence $\langle X_0, \cdots, X_n\rangle$ containing all P-solutions or there is an accumulation point Y of the class $\{X : P(X)\}$, i.e., every neighborhood of Y contains two X such that $P(X)$.*

Definition 6 (Π^1_1-SEP)**.** *Given two Π^1_1 predicates $\phi(n), \psi(n)$,*

$$\forall n\, \neg\phi(n) \vee \neg\psi(n) \to \exists X\, \forall n\, (\phi(n) \to n \in X) \wedge (\psi(n) \to n \notin X).$$

Definition 7 (Δ^1_1-CA)**.** *Given a Π^1_1 predicate $\phi(n)$ and a Σ^1_1 predicate $\psi(n)$,*

$$\forall n\, \phi(n) \leftrightarrow \psi(n) \to \exists X\, \forall n\, \phi(n) \leftrightarrow n \in X.$$

Below zoo of THA is taken from BGS [2]. Also have a look at [3].

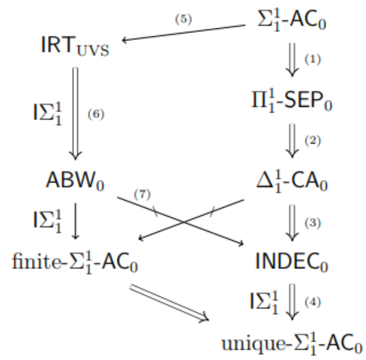

Figure 4: Partial zoo of theories of hyperarithmetic analysis. Single arrows indicate implication while double arrows indicate strict implication. The references for the above results are as follows: (1, 2) Montalbán [17, Theorems 2.1, 3.1]; (3, 4) Montalbán [16, Theorem 2.2], Neeman [19, Theorems 1.2, 1.3, 1.4], see also Neeman [20, Theorem 1.1]; (5) Theorem 4.5; (6) Theorems 7.7, 7.10; (7) Conidis [4, Theorem 4.1]. All results concerning finite-Σ^1_1-AC$_0$ are in Goh [10].

Lemma 8 (RCA$_0$)**.** IRT$^-$ $\to$ ACA

Theorem 9 (RCA$_0$ + IΣ^1_1)**.** IRT$^-$ $\to$ ABW

Proof. ($\to$) Assume IRT$^-$. Let $P(X)$ be an arithmetic predicate on $2^{\mathbb{N}}$ which does not hold finitely many solutions. Since $P(X)$ does not hold finitely many solutions, given n distinct P-solutions $X_0, \ldots, X_{n-1}$, there exists an X_n distinct from $X_0, \ldots, X_{n-1}$ such that $P(X_n)$. Inductively, $P(X)$ holds for arbitrarily many solutions (IΣ^1_1). Let T be a binary tree so that an $X \in [T]$ iff $P(X)$ (ACA). View T as a graph. Given an arbitrary sequence $X_0, \ldots, X_{n-1}$ of distinct P-solutions. Take sufficiently large m so that $(X_0)_{\geq m}, (X_1)_{\geq m}, \ldots, (X_n)_{\geq m}$ are disjoint. Therefore, T contains arbitrarily many disjoint rays. Invoke IRT$^-$ to obtain an infinite sequence of disjoint subgraphs $\langle G_n \rangle_n$ each of which contains a ray. Without loss of generality, assume these G_n are connected (ACA). Now viewing G_n as filters in the tree T, take the minimum elements g_n from each G_n. Take the downward closure G of $(g_n)_n$ in the tree T, and invoke WKL to obtain a path $Z \in [G]$. This Z is the desired accumulation point for $P(X)$. $\square$

2 Zero type choice principle

Definition 10. Σ^1_1-AC *is the axiom schema:*

$$\forall n \exists X\, \phi(n, X) \to \exists Y \forall n\, \phi(n, Y^{[n]})$$

consisting of all Σ^1_1 *formulas* $\phi(n, X)$ *that does not contain* Y *as a free variable, where* $Y^{[n]} = \{m : \langle n, m \rangle \in Y\}$ *means the* n*th column of* Y*.*

Definition 11. Σ^1_1-AC^0 *is the axiom schema:*

$$\forall n \exists i\, \phi(n,i) \to \exists f \forall n\, \phi(n, f(n))$$

consisting of all Σ^1_1 formulas $\phi(n,i)$ that does not contain f as a free variable.

Definition 12. Σ^1_1-$\mathrm{AC}^{0,k}$ *is the axiom schema:*

$$\forall n \Big(\exists i \phi(n,i) \land \forall i_0, \cdots, i_k \big(i_0, \cdots, i_k \text{ distinct} \to \neg\phi(n,i_0) \lor \cdots \lor \neg\phi(n,i_k) \big) \Big) \\ \to \exists f \forall n\, \phi(n, f(n))$$

consisting of all Σ^1_1 formulas $\phi(n,i)$ that does not contain f as a free variable.

Theorem 13 (RCA_0)**.** Σ^1_1-$\mathrm{AC}^{0,1} \leftrightarrow \Delta^1_1$-$\mathrm{CA}$

Proof. ($\to$) Assume Σ^1_1-$\mathrm{AC}^{0,1}$. Suppose two Π^1_1 formulas $\phi(n), \psi(n)$ satisfy $\forall n\, (\phi(n) \leftrightarrow \neg\psi(n))$. Consider the Σ^1_1 formula $A(n,i) := (\phi(n) \to i = 0) \land (\psi(n) \to i = 1)$. For each n, there exists a unique i that satisfy $A(n,i)$. Invoke Σ^1_1-$\mathrm{AC}^{0,1}$ to obtain an $f : \mathbb{N} \to \{0,1\}$ such that $\forall n\, A(n, f(n))$. Let $X = \{n : f(n) = 0\}$. Then, $n \in X$ iff $\phi(n)$.

($\leftarrow$) Assume Δ^1_1-CA. Suppose a Σ^1_1 formula $\phi(n,i)$ holds for exactly one i to each n. Consider the Π^1_1 formula $\psi(n,i) := \forall j (j \neq i \to \neg\phi(n,j))$. Then, $\phi(n,i) \leftrightarrow \psi(n,i)$. Invoke Δ^1_1-CA to obtain $X = \{\langle n,i \rangle : \phi(n,i)\}$. The set X is the graph of the desired $f : \mathbb{N} \to \mathbb{N}$ satisfying $\forall n\, \phi(n, f(n))$. $\square$

Theorem 14 (RCA_0)**.** Σ^1_1-$\mathrm{AC}^{0,k} \leftrightarrow \Pi^1_1$-$\mathrm{SEP}$ ($k \geq 2$)

Proof. (Σ^1_1-$\mathrm{AC}^{0,2} \to \Pi^1_1$-$\mathrm{SEP}$) Assume Σ^1_1-$\mathrm{AC}^{0,2}$. Suppose two Π^1_1 formulas $\phi(n), \psi(n)$ satisfy $\forall n\, (\neg\phi(n) \lor \neg\psi(n))$. Consider the Σ^1_1 formula $A(n,i) := (\phi(n) \to i = 0) \land (\psi(n) \to i = 1) \land (i = 0 \lor i = 1)$. For each n, there exists an i, and at most two possible i's that satisfy $A(n,i)$. Invoke Σ^1_1-$\mathrm{AC}^{0,2}$ to obtain an $f : \mathbb{N} \to \{0,1\}$ such that $\forall n\, A(n, f(n))$. Let $X = \{n : f(n) = 0\}$. Then, X separates ϕ and ψ.

(Π^1_1-$\mathrm{SEP} \to \Sigma^1_1$-$\mathrm{AC}^{0,k}$) Assume Π^1_1-SEP. Suppose a Σ^1_1 formula $\phi(n,i)$ holds for at least one but at most $k \geq 2$ number of i's for each n. Consider the two Π^1_1 formulas $\psi_1(n,i) := \forall j (j \neq i \to \neg\phi(n,j))$ and $\psi_2(n,i) := \neg\phi(n,i)$. Notice ψ_1, ψ_2 are disjoint: $\forall n, i\, (\neg\psi_1(n,i) \lor \neg\psi_2(n,i))$. Invoke Π^1_1-SEP to obtain $X \supseteq \{\langle n,i \rangle : \psi_1(n,i)\}$ such that $X^c \supseteq \{\langle n,i \rangle : \psi_2(n,i)\}$. If $\langle n,i \rangle \in X$, then $\phi(n,i)$ must hold since $X^c \supseteq \{\langle n,i \rangle : \neg\phi(n,i)\}$. So, consider the new Π^1_1 formula $\psi'_1(n,i) := \forall j (\langle n,j \rangle \notin X) \land \exists i' \forall j\, (j \neq i \land j \neq i' \to \neg\phi(n,j))$. It must be that ψ'_1 and ψ_2 are disjoint since $\forall j (\langle n,j \rangle \notin X)$ only holds when $\phi(n,i)$ holds for more than one i for a fixed n. Invoke again Π^1_1-SEP to obtain $X' \supseteq \{\langle n,i \rangle : \psi'_1(n,i)\}$ such that $(X')^c \supseteq \{\langle n,i \rangle : \psi_2(n,i)\}$. Inductively (meta induction outside of Π^1_1-SEP_0), let $\psi^{(l)}_1(n,i) := \forall j (\langle n,j \rangle \notin X \cup \cdots \cup X^{(l-1)}) \land \exists i', \cdots, i^{(l)} \forall j\, (j \neq i \land j \neq i' \land \cdots \land j \neq i^{(l)} \to \neg\phi(n,j))$ to obtain $X^{(l)}$ for $1 \leq l \leq k-1$. Define $f(n) = i$ for the lexicographically least $\langle l,i \rangle \in \{0, \cdots, k-1\} \times \mathbb{N}$ such that $\langle n,i \rangle \in X^{(l)}$. Then, $\forall n\, \phi(n, f(n))$. $\square$

References

[1] James S. Barnes, Jun Le Goh, and Richard A. Shore. Theorems of hyperarithmetic analysis and almost theorems of hyperarithmetic analysis. *The Bulletin of Symbolic Logic*, 28(1):133–149, 2022.

[2] James S. Barnes, Jun Le Goh, and Richard A. Shore. Halin's infinite ray theorems: Complexity and reverse mathematics. *Journal of Mathematical Logic*, 25(01):2450010, 2025.

[3] Antonio Montalbán. On the π11-separation principle. *Mathematical Logic Quarterly*, 54:563 – 578, 11 2008.