CCR 2025 Schedule with Abstracts

Monday, June 16

8:45–9:30 Coffee and registration

9:30–10:00 Welcome address

10:00–11:00 Joseph S. Miller – *Randomness and the enumeration degrees*

I will discuss several results and questions that emerge from the unholy marriage of measure/randomness and the enumeration degrees. We will start with a proof that all weakly 2-random sets have quasiminimal enumeration degree. We will finish with questions about the continuous degrees and about K-triviality that arise when we relativize Martin-Löf randomness to enumeration oracles.

11:00–11:45 Takayuki Kihara – *Computability-Theoretic Perspectives on the Katetov order*

The notion of ideal (dually, filter) on sets play an important role in various fields including computability theory. There are countless important examples of ideals on sets: Ideals of Lebesgue null sets, meager sets, summable sets, sets with asymptotic density zero, etc.

In set theory and related areas, as a method for comparing the complexity of ultrafilters, the Rudin-Keisler order $\leq_{RK}$ *is well-studied, and as a method for comparing the complexity of ideals, the Katětov order* $\leq_K$ *is well-studied. For ultrafilters, the Rudin-Keisler order coincides with the Katětov order (for dual ideals). In this sense, the Katětov order has a broader scope.*

Our new computability-theoretic perspective is: The Katětov order is Weihrauch reducibility for ideals.

Here, Weihrauch reducibility is a notion actively studied in computable analysis to measure the computability-theoretic complexity of Π_2 *theorems. What does this new perspective bring? For example, it becomes clear that the Fubini product* $\otimes$ *of ideals (filters) is precisely the compositional product* $\star$ *in the context of Weihrauch reducibility. In this way, we can provide explanations of several notions in the Katětov order of ideals from the computability-theoretic perspective, and conversely, we can apply the ideas of the Katětov order to computability theory. This is joint work with Ming Ng.*

11:45–13:45 Lunch & coffee

13:45–14:45 Emmanuel Rauzy – *Effective bases and notions of effective second countability in computable analysis*

We investigate different notions of "computable topological basis" for represented spaces. We show that several non-equivalent notions of bases become equivalent when we consider computably enumerable bases. This indicates the existence of a robust notion of computably second countable represented space. These *spaces are precisely those introduced by Grubba and Weihrauch under the name "computable topological spaces". The present work thus clarifies the articulation between Schröder's approach to computable topology based on the Sierpinski representation and other approaches based on notions of computable bases. These other approaches turn out to be compatible with the Sierpinski representation approach, but also strictly less general.*

We revisit Schröder's Effective Metrization Theorem, by showing that it characterizes those represented spaces that embed into computable metric spaces: those are the computably second countable strongly computably regular represented spaces.

Finally, we study different forms of Open Choice problems. We show that having a computable open choice is equivalent to being computably separable, but that the "non-total open choice problem", i.e., Open Choice restricted to open sets that have non-empty complement, interacts with effective second countability in a satisfying way.

14:45–15:30 Ivan Titov – *Martin-Löf randomness and differentiation*

Computable randomness of a real can be characterized as the differentiability of every nondecreasing (Brattka, Miller, Nies, [DOI:10.1090/tran/6484]) or Lipschitz continuous (Freer, Kjos-Hanssen, Nies, Stephan, [DOI:10.3233/COM-14025]) computable (in the sense of computable analysis) function in this real.

In the current work, we prove that, on the set of left-c.e. reals, Martin-Löf randomness can be characterized as left-differentiability of every nondecreasing (or Lipschitz continuous) function, which is computable left from this real, and extend this result to all reals.

15:30–16:00 Coffee

16:00–16:45 Thibaut Kouptchinsky – *The reverse mathematics of analytic measurability*

This talk is about a work in progress with Juan Aguilera (TU Wien) and Keita Yokohama (Tohoku University) on the foundations of mathematics, studying measurability of analytic sets, with the method of random forcing.

We study a theorem of Lusin, which stated that all analytical set of reals are Lebesgue-measurable. We work in the framework of second-order arithmetic. Yu showed that assuming transfinite recursion is sufficient (and necessary) to prove that all Borel sets are Lebesgue-measurable, coded as wellfounded trees on natural numbers. Simpson asked whether the samer assumption "suffices to prove measurability and regularity of analytic sets in some appropriate sense." The purpose of our work is to answer this question.

We show that Lebesgue-regularity, i.e., the equality of the outer and inner measures is equivalent to induction for formula with one existential set quantifier; a substytem of second order arithmetic intermediate between transfinite recursion and comprehension for the same class of formula (i.e. existence of analytical sets). This is the first reversal of the kind we know of in reverse mathematics.

If one additionally demands that this value exist as a number, the strength of Lusin's theorem increases to the previously mentionned comprehension system.

The main idea is to draw inspiration from Solovay's construction of a model of Zermelo-Fraenkel set theory where every set is Lebesgue measurable. In our case the forcing argument is carried out over a non-standard model of a weak set theory (obtained through the familiar method of pseudohierarchies).

Tuesday, June 17

8:45–9:15 Coffee

9:15–10:15 Kenshi Miyabe – *Solovay reducibility for computably approximable reals*

I will give a survey of recent developments in Solovay reducibility for computably approximable reals, focusing on connections to analytical concepts and their structural properties.

10:15–11:00 Akhil S. – *Resource-Bounded Kučera-Gács Theorems*

The classical Kučera-Gács Theorem states that every infinite binary sequence is Turing reducible from a Martin-Löf random sequence. Thus, every sequence can be seen as the result of a random source being transformed by a deterministic algorithm with unbounded computational power. In this ongoing work with Satyadev Nandakumar and Chandra Shekhar Tiwari, we explore how this result behaves under computational resource bounds.

We first prove an almost polynomial-time analogue of the Kučera-Gács Theorem. For every sequence X, there exists a polynomial-time random sequence R such that X is computable from R using a super-polynomial time reduction. This is achieved via a construction that diagonalizes against all polynomial-time martingales using a universal super-polynomial-time martingale, ensuring that R remains random while encoding enough information to reconstruct X in polynomial-time.

Doty (CiE 2006) showed that the Kučera-Gács theorem has key implications for constructive dimension, linking the efficiency of extraction from randomness to the information density of a sequence. Building on this viewpoint, we turn to questions about polynomial-time decompression and dimension. We show that the lower polynomial-time decompression ratio $\rho^-_{poly}(X)$ is equal to the polynomial-time Kolmogorov complexity rates $K_{poly}(X)$, resolving a gap in prior work. Combined with recent results showing $K_{poly} \neq \mathrm{cdim}_P$ assuming one-way functions, we deduce that $\rho^-_{poly} \neq \mathrm{cdim}_P$ under the same assumption.

Lastly, we investigate whether Kučera-Gács-type results extend to finite-state reductions. We show that they do not: there exist sequences which cannot be computed from any normal (finite-state random) sequence using a finite-state transducer.

11:00–11:45 Quentin Le Houérou – *Ramsey-like theorems for the Schreier barrier*

The family of finite subsets s of the natural numbers such that $|s| = 1 + \min s$ is known as the Schreier barrier in combinatorics and Banach Space theory, and as the family of exactly ω-large sets in Logic.

Instead of considering the infinite Ramsey's theorem for colorings of pairs, triplets or more generally n-uplets, we can study colorings of the exactly ω-large sets. The resulting statement is true but very strong from a computational and reverse mathematical point of view.

Similarly, in this presentation we will look at the generalizations to colorings of the exactly ω-large sets of other (Ramsey-like) theorems such as Friedman's Free Set and Thin Set theorems and of the Rainbow Ramsey's theorem. Surprisingly, although all these theorems have a low computational power in the case of coloring of n-uplets (none of them being able to compute the halting set), their exactly ω-large counterparts exhibits different behaviors, with the Thin Set and Free Set theorems versions being able to encode the ω-th iteration of the Turing jump, while the Rainbow Ramsey's theorem version still does not code the halting set.

This is a joint work with Lorenzo Carlucci, Oriola Gjetaj and Ludovic Levy Patey.

11:45–13:45 Lunch & coffee

13:45–14:45 Satyadev Nandakumar – *The non-robustness of polynomial-time dimension and the existence of one-way functions*

(joint work with Subin Pulari (Bordeaux), Akhil S (IIT Kanpur) and Suronjona Sarma (IIT Kanpur))

We show a duality between the non-robustness of polynomial time dimension and the existence of one-way functions. Polynomial-time dimension (denoted $\mathrm{cdim}_{\mathrm{P}}$*) quantifies the density of information of infinite sequences using polynomial time betting algorithms called s-gales. An alternate quantification of the notion of uses polynomial-time Kolmogorov complexity rate (denoted* $\mathcal{K}_{poly}$*). Hitchcock and Vinodchandran (CCC 2004) showed that* $\mathrm{cdim}_{\mathrm{P}}$ *is at least as large as* $\mathcal{K}_{poly}$*. We first show that if one-way functions exist then there exists a polynomial-time samplable distribution with respect to which* $\mathrm{cdim}_{\mathrm{P}}$ *and* $\mathcal{K}_{poly}$ *are separated by a uniform gap with probability* 1*. Conversely, we show that if there exists such a polynomial-time samplable distribution, then (infinitely-often) one-way functions exist.*

Using our main results, we solve a long standing open problem posed by Hitchcock and Vinodchandran (CCC 2004) and Stull under the assumption that one-way functions exist. We demonstrate that if one-way functions exist, then there are individual sequences X *whose poly-time dimension strictly exceeds* $\mathcal{K}_{poly}(X)$*, that is* $\mathrm{cdim}_{\mathrm{P}}(X) > \mathcal{K}_{poly}(X)$*. The corresponding unbounded notions, namely, the constructive dimension and the asymptotic lower rate of unbounded Kolmogorov complexity are equal for every sequence. Analogous notions are equal even at polynomial space and finite-state levels. In view of these results, it is reasonable to conjecture that the polynomial-time quantities are identical for every sequence and set of sequences. However, under a plausible assumption which underlies modern cryptography - namely the existence of one-way functions, we* refute *the conjecture thereby giving a negative answer to the open question posed by Hitchcock, Vinodchandran and Stull. Further, we show that the gap between these quantities can be made as large as possible (i.e. close to 1). We also establish similar bounds for strong poly-time dimension versus asymptotic upper Kolmogorov complexity rates.*

Our proof uses several new constructions and arguments involving probabilistic tools such as the Borel-Cantelli Lemma, the Kolmogorov inequality for martingales and the theorem on universal extrapolation by Ilango, Ren, and Santhanam. This work shows that the question of non-robustness of polynomial-time information density notions, which is prima facie *different, is intimately related to questions which are of current interest in cryptography and meta-complexity.*

14:45–15:30 Mariya Soskova – *Posner-Robinson theorem in the enumeration degrees*

Posner and Robinson proved that if A *is a collection of Turing degrees uniformly computable in* $\mathbf{0}$ *then there is a low degree that joins all of them to* $\mathbf{0}$*. Shore and Slaman extend this result for the case when* A *has size* 1 *to higher iterations of the jump and use this to prove the definability of the Turing jump. We explore statements of this form in the enumeration degrees for the enumeration jump operator and the enumeration skip operator.*

Enumeration reducibility is a positive reducibility between sets of natural numbers, whose introduction was motivated by the wish to extend Turing reducibility to partial functions in a meaningful way. We say that X *is enumeration reducible to* Y *if every enumeration of* Y *computes an enumeration of* X*. The Turing degrees embed into the enumeration degrees as the total degrees. There are two known ways to define an analog of the Turing jump operator: the skip and the enumeration jump both agree with the Turing jump restricted to total degrees but behave differently on non-total degrees. The enumeration jump of the degree* $\mathbf{a}$ *is the least upper bound of the skip of* $\mathbf{a}$ *and* $\mathbf{a}$*.*

Kalimullin showed that Posner and Robinson's theorem for sets A *of size* ≥ 2 *fails in the enumeration degrees for the jump operator because of the existence of a special kind of pairs of enumeration degrees later called K-pairs (in his honor). Using ideas from work by Gura we show that this is the only obstacle. The situation with the skip operator is much more intriguing. On the one hand the skip of* $\mathbf{a}$ *is not above* $\mathbf{a}$*. On the other hand the double skip of* $\mathbf{a}$ *can sometimes be equal to* $\mathbf{a}$*. We present joint work with Slaman, in which we discover an unexpected failure of relativized skip inversion and a way to avoid it using K-pairs to prove that for every enumeration degree* $\mathbf{x}$ *there is an enumeration degree* $\mathbf{g}$ *such that* *the least upper bound of a and x equals the least upper bound of the skip and the double skip of g.*

15:30–16:00 Coffee

16:00–16:45 Alexis Terrassin – *Complexité descriptive de certaines propriétés topologiques d'espaces compacts métrisables*

Dans cette présentation, je m'intéresserai à la complexité descriptive de certaines propriétés topologiques d'espaces compacts métrisables, dans le cadre de leur classification à homéomorphisme près. En m'appuyant sur les outils de la théorie descriptive des ensembles et sur l'hyperespace des compacts, je montrerai comment on peut mesurer la complexité borélienne de propriétés comme la connexité, la connexité locale, ou encore le fait d'être homéomorphe à un graphe, un disque, ou une surface. Un des points clés est l'utilisation de la notion d'imitation topologique, qui permet de capturer des invariants avec une complexité bien maîtrisée. J'aborderai notamment plusieurs résultats de Π^0_3-complétude et je discuterai de la classification des surfaces fermées orientables.

Wednesday, June 18

8:45–9:15 Coffee

9:15–10:15 Rahul Santhanam – *Meta-Complexity: Introduction and a brief survey*

Meta-complexity is the study of the complexity of computational problems that are themselves about complexity, e.g., the Minimum Circuit Size Problem (does a Boolean function F given by its truth table have size at most s) and the problem of computing time-bounded Kolmogorov complexity. I will begin my motivating meta-complexity and defining the basic notions. I will then sketch applications in learning, cryptography and complexity lower bounds.

10:15–11:00 Marius Zimand – *On one-way functions and the average time complexity of almost-optimal compression*

We show that one-way functions exist if and only if there exists an efficiently-samplable distribution relative to which almost-optimal compression is hard on average. The result is obtained by combining a theorem of Ilango, Ren, and Santhanam [STOC 2022] and one by Bauwens and Zimand [J ACM 2023].

11:00–11:30 Coffee

11:30–12:15 Ellen Hammatt – *An introduction to structures computable without delay*

The study of punctual structures considers what happens to the computation of presentations and their isomorphisms when we forbid the use of unbounded search, i.e., the restriction to primitive recursive computations. In 2017, Kalimullin, Melnikov and Ng initiated the study of punctual structures, that is, structures that are computable without delay. In this talk we will give an introduction to this area, focusing on the new techniques and ideas that come up compared to that of computable structure theory. We will discuss the motivations to study this area and some of the interesting results that have been collected so far.

12:15–16:15 Lunch & Free time

16:15–18:00 Social program

18:00–... Conference dinner

Thursday, June 19

8:45–9:15 Coffee

9:15–10:15 Andrea Sorbi – *Enumeration reducibility and singleton reducibility: Three old projects of Barry Cooper*

The following three theorems answer old questions raised by Barry Cooper and other early scholars working in the field of positive reducibilities.

In 1996 Barry and I proved that there exists an incomplete Σ^0_2 enumeration degree which is noncappable, i.e. an enumeration degree strictly below the first jump which is not half of any minimal pair. The proof seemed to suggest that any incomplete noncappable enumeration degree should be properly Σ^0_2, i.e. should contain no Δ^0_2 set. For many years, Barry and I unsuccessfully tried to verify this conjecture. Eventually (jointly with Keng Meng Ng) we are able to provide the expected answer.

Theorem 1 [Cooper, Ng, S., Yang] Every incomplete Δ^0_2 enumeration degree is cappable.

The next two theorems deal with an important and interesting subreducibility of enumeration reducibility, called singleton reducibility. Theorem 2 answers a question originally raised by Polyakov and Rozinas (1977) , and repeatedly re-proposed by other scholars including Zacharov (1984), Cooper (1990), Bathyrshin (2017).

Theorem 2 [Kent, Ng, S.] Every nonzero enumeration degree contains infinitely many singleton degrees. In fact no nonzero enumeration degree contains a minimal singleton degree.

Finally, Theorem 3 answers a question raised by Cooper (1987, 1990).

Theorem 3 [Kent, Ng, S.] The Σ^0_2 singleton degrees are not dense. In fact there exist two Δ^0_2 sets B, C lying in the same enumeration degree, such that the singleton degree of C is a minimal cover of the singleton degree of B.

10:15–11:00 Subin Pulari – *The Normality-Dimension thesis: Recent progress and future directions*

Finite-state dimension—a finite-state analogue of classical Hausdorff dimension—quantifies the density of information in an infinite sequence over a finite alphabet, as measured using finite-state automata. The Schnorr-Stimm dichotomy theorem (1972) gives the following important connection between dimension and normality - an infinite sequence has finite-state dimension equal to 1 if and only if it is normal. Many classical results about normal sequences—i.e. sequences having finite-state dimension 1—have been shown to be special cases of more general theorems concerning finite-state dimension. In his talk at CCR 2022, Jack Lutz proposed the following hypothesis - every theorem about Borel normality is a special case of a theorem about finite-state dimension. In this talk, we revisit the Normality-Dimension Thesis and review the validity of the hypothesis in the context of known results involving finite-state dimension, which generalizes theorems concerning normality. Thereafter, we explore some very recent progress that supports the thesis obtained by the author in joint works with Bienvenu, Gimbert and Nandakumar. We conclude by highlighting directions for future research concerning the validity and implications of Lutz's Normality-Dimension Thesis.

11:00–11:45 Diego Rojas – *Algorithmic randomness in harmonic analysis*

Within the last fifteen years, there has been an ongoing program to use almost-everywhere theorems in analysis and ergodic theory to study algorithmic randomness. In harmonic analysis, Franklin, McNicholl, and Rute characterized Schnorr randomness using an effective version of Carleson's Theorem. We show here that, for computable $1 < p < \infty$, the reals at which the Fourier series of a weakly computable vector in $L^p[-\pi, \pi]$ converges are precisely the Martin-Löf random reals. Furthermore, we show that radial limits of the Poisson integral of an $L^1(R)$-computable function coincide with the values of the function at exactly the Schnorr random reals.

11:45–13:45 Lunch & coffee

13:45–14:30 Kakeru Yokoyama – *Randomness and agnostic online learning*

In this talk, we explore a new connection between Machine Learning and Algorithmic Randomness. An important field in Machine Learning is online learning, which has the following setup. A learner is presented with a sequence of data instances, one at a time. Based on the previous data, the learner attempts to make an accurate prediction for what the next data instance will be.

There is a striking parallel between this setup and the martingale characterisation of Martin-Löf Randomness: namely, a binary sequence is said to be ML-random if no lower semicomputable martingale succeeds on it. Both settings feature a learner presented with a finite sequence of data points, but use different notions of what it means for the learner to make "good" predictions. In agnostic online learning, one asks if the learning algorithm minimises regret with respect to a given set of experts. In algorithmic randomness, one asks if the martingale is able to make infinite capital – a criterion for determining the complexity of the binary sequence.

We propose to view the class of sequences that cannot be predicted accurately by agnostic c-online learning as a form of randomness class, and discuss how it compares with existing notions of randomness. For this comparison, we consider several approaches, including one based on the properties of martingales corresponding to online learning algorithms, and another that constructs online prediction algorithms from randomness tests.

This is joint work with Ming Ng.

14:30–15:15 Miguel Aguilar Enriquez – *Reverse mathematics of the mountain pass theorem*

(The present project is part of my ongoing research towards a Doktorat at TU Wien)

The main goal of this work in progress is to prove that the Mountain Pass Theorem (in short MPT) of Ambrosetti and Rabinowitz is equivalent to WKL0 over RCA0 in the framework of the research program of Reverse Mathematics. Broadly speaking, the MPT provides necessary conditions to ensure the existence of critical points of differentiable functionals with domain defined in a Hilbert space and image in the real numbers.

In order to prove that WKL0 implies the MPT over RCA0, we develop some Analysis within WKL0 to have access to the space of continuous functions from [0,1] into a separable Banach space and from there built formalized proofs of the basic ingredients of the Mountain Pass Theorem: The deformation lemma and the minimax principle that proves the theorem itself. A dive in the theory of Ordinary Differential Equations is also nedded and interesting by itself.

For the other direction, to prove that the MPT implies WKL0 over RCA0, we use the contrapositive and assuming the existence of a infinite binary tree with no path, we computably construct a smooth function satisfying all the hypotheses of the MPT but not its conclusion.

15:15–15:45 Coffee

15:45–16:30 Kohtaro Tadaki – *A refinement of the theory of quantum error-correction by algorithmic randomness*

The notion of probability plays a crucial role in quantum mechanics. It appears in quantum mechanics as the Born rule. In modern mathematics which describes quantum mechanics, however, probability theory means nothing other than measure theory, and therefore any operational characterization of the notion of probability is still missing in quantum mechanics. In our former works [K. Tadaki, arXiv:1804.10174], based on the toolkit of algorithmic randomness, we presented a refinement of the Born rule, called the principle of typicality, for specifying the property of results of measurements in an operational way. In this talk, we make an application of our framework to the theory of quantum error-correction for refining it, in order to demonstrate how properly our framework works in practical problems in quantum mechanics.

Friday, June 20

8:45–9:15 Coffee

9:15–10:15 Elvira Mayordomo – *Using Information Theory in geometric measure theory and profinite groups*

Effective and resource-bounded dimensions were defined by J. Lutz and have proven to be useful and meaningful for quantitative analysis in the contexts of algorithmic randomness, computational complexity and fractal geometry.

The point-to-set principle (PSP) of J. Lutz and N. Lutz fully characterizes Hausdorff dimensions in terms of effective dimensions in the Euclidean space, enabling effective dimensions to be used to answer open questions about fractal geometry, with already an interesting list of geometric measure theory results. PSP has been recently extended to any separable metric space.

In this talk I will review the point-to-set principles and present very recent applications to both geometric measure theory and profinite groups. These recent applications are joint work with P. Cholak, M. Csornyei, N. Lutz, P. Lutz, D. M. Stull and with A. Nies respectively.

10:15–11:00 Rui Li – *Bounded arithmetic and differential equations*

An ODE (ordinary differential equation) scheme has been proposed to characterize FP, the class of functions computable in polynomial time. This new approach of program-via-ODE paradigm has since generalized to other deterministic complexity classes, such as FPSPACE and FAC0. However, we still lack the mechanism of generalizing it to nondeterministic classes, such as TFNP. We hereby introduces one possible approach to the ODE characterization of the complexity of nondeterministic functions.

11:30–12:15 Jack Yoon – *Type zero choice principle and Halin's infinite ray theorem*

Halin's infinite ray theorem (IRT) states that a graph with arbitrarily many disjoint rays contains infinitely many disjoint rays. It has been investigated as a reverse mathematical principle by Barnes, Goh, and Shore (BGS). Various versions of IRT have been shown to be theorems of hyperarithmetic analysis (THA). This THA family houses principles of varying strengths including the arithmetical transfinite recursion ATR *of the big five, the choice principle* Σ^1_1-AC, *or the arithmetic Bolzano-Weierstrass theorem* ABW. *Among many versions of the infinite ray theorem, we will discuss the most approachable variant* $\mathrm{IRT}_{\mathrm{UVS}}$, *which has been placed between* Σ^1_1-AC *and* ABW, *with some help from the induction schema* $\mathrm{I}\Sigma^1_1$. *We find it appropriate to introduce a weakening* $\mathrm{IRT}^-_{\mathrm{UVS}}$, *by formalizing the infiniteness of a collection of sets differently.*

Given a predicate $P(X)$ with a set variable X, infiniteness of P-solutions can be asserted in at least two different ways:

1. *There is an infinite distinct collection $(Y_0, Y_1, ...)$ of P-solutions.*
2. *There exists an infinite collection of disjoint sets $(X_0, X_1, \ldots, X_n, \ldots)$, each of which contains a subset $Y_n \subseteq X_n$ satisfying $P(Y_n)$.*

The statement of $\mathrm{IRT}_{\mathrm{UVS}}$ *as formalized by BGS uses the first, and our weakening* $\mathrm{IRT}^{0}_{\mathrm{UVS}}$ *uses the second. At a first glance, disjoint requirements for the sets $(X_0, X_1, \ldots, X_n, \ldots)$ make the second assertion seem stronger, but in the context of* $\mathrm{IRT}^{0}_{\mathrm{UVS}}$*, we will see that, along with the preexisting disjoint requirements for the rays, the inability to locate Y_n's uniformly makes it weaker* ($\mathrm{RCA}_0 \vdash \mathrm{IRT}_{\mathrm{UVS}} \rightarrow \mathrm{IRT}^{-}_{\mathrm{UVS}}$).

To complement the introduction of $\mathrm{IRT}^{-}_{\mathrm{UVS}}$*, we also introduce a type zero choice principle* $\Sigma^1_1\text{-AC}^0$*, which asserts choice over numbers rather than sets. This type zero choice is weaker than the usual choice* ($\mathrm{RCA}_0 \vdash \Sigma^1_1{-}\mathrm{AC} \rightarrow \Sigma^1_1{-}\mathrm{AC}^0$)*. Surprisingly, we find that further restrictions* $\Sigma^1_1{-}\mathrm{AC}^{0,k}$ *by allowing at most k choices connect to Montalbán's chain* $\Sigma^1_1{-}\mathrm{AC}_0 \Rightarrow \Pi^1_1{-}\mathrm{SEP}_0 \Rightarrow \Delta^1_1{-}\mathrm{CA}_0$ *in the following way: over* RCA_0*, we prove* $\Sigma^1_1{-}\mathrm{AC}^{0,1}$ *is equivalent to the comprehension axiom* $\Delta^1_1\text{-CA}$*, and* $\Sigma^1_1{-}\mathrm{AC}^{0,k}$ *for each $k \geq 2$ is equivalent to the separation principle* $\Pi^1_1{-}\mathrm{SEP}$.

We will discuss how these new results provide us an updated knowledge about the location of IRT.