Computability-Theoretic Perspectives on the Katětov order

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Two directions for *"relaxed"* computability:

1. Allow to use *an oracle* during a computation.
2. Allow *a few mistakes* during a computation.

- The first direction has been well-studied:
  That is, Turing, and other degree theory.
- For the second direction, there have been several isolated studies
  ... but a well-established unified theory does not seem to exist yet.

What does it mean to allow *a few mistakes*?

- *Example 1.* Fix $n, k \in \mathbb{N}$ with $\frac{k}{n} \gg 1/2$.
  Run n many computations simultaneously,
  and if k of them are successful, then it is OK.
- *Example 2.* Probabilistic computation:
  Perform a computation correctly with probability $1 - \varepsilon$
  (a computation error with probability ε is OK).

In other words, it is computability by *"majority."*

- Run multiple computations simultaneously;
  then adopt the computation results of the "*majority*."

What is ... *majority*, in mathematical terms?

- An *(ultra-)filter*, or simply, an *upper set* $\mathcal{U}$ on a set I.
  — $A \in \mathcal{U}$ and $A \subseteq B \subseteq I \implies B \in \mathcal{U}$.

Let $\mathcal{U}$ be an (ultra-)filter on a set I.

- Run I-many computations $(f_i)_{i \in I}$ simultaneously; then adopt the computation results $(f_i)_{i \in A}$ of a *majority* $A \in \mathcal{U}$.

Example.

The set of all Lebesgue conull sets $A \subseteq \mathbf{2}^{\mathbb{N}}$ form a filter on $\mathbf{2}^{\mathbb{N}}$.

- This yields a kind of randomized computability.

How to compare "*majority notions*."

- The Rudin-Keisler order is a well-known method for comparing the complexity of ultrafilters.
- The Katětov order is a well-known method for comparing the complexity of ideals (dually, filters).

▷ (Fact) The Rudin-Keisler order and the Katětov order coincide for ultrafilters. So, we consider only the more general Katětov order.

Recall: Two directions for "*relaxed*" computability:

- *Oracle:* Computability using a (multi-valued) oracle (compared, for example, using Weihrauch reducibility)
- *Majority:* Computability that allows a few errors (compared, for example, using Katětov order)

Key Claim: Weihrauch reducibility and Katětov order are integrated by a common underlying notion.

Let $F, G :\subseteq \mathbb{N} \rightrightarrows \mathbb{N}$.

Definition. Weihrauch reducibility $F \leq_W G$:

- given an instance $x \in \mathbf{dom}(F)$,
- an inner reduction φ yields an instance $\tilde{x} \in \mathbf{dom}(G)$,
- and then, given a solution $y \in G(\tilde{x})$,
- an outer reduction ψ yields a solution $\tilde{y} \in F(x)$.

	Player I	*Player II*
1:	$x \in \mathbf{dom}(F)$	
2:		$\tilde{x} \in \mathbf{dom}(G)$
3:	$y \in G(\tilde{x})$	
4:		$\tilde{y} \in F(x)$

▷ A value of $F(x)$ can be computed by making a *single query* to the oracle G.

Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{P}(\mathbb{N})$ be upper sets.

Definition. Katětov reducibility $\mathcal{U} \leq_K \mathcal{V}$:

- given a majority $A \in \mathcal{U}$,
- there exists a majority $B \in \mathcal{V}$,
- and then, given any member x of the majority B,
- a reduction ψ yields a member $y \in A$.

	Player I	Player II
1:	$A \in \mathcal{U}$	
2:		$B \in \mathcal{V}$
3:	$x \in B$	
4:		$y \in A$

▷ Equivalently, $A \in \mathcal{U} \implies \psi^{-1}[A] \in \mathcal{V}$.

Let $F :\subseteq \mathbb{N} \rightrightarrows \mathbb{N}$; let $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ be an upper set.

Hybrid. Computability by majority $F \leq \mathcal{U}$:

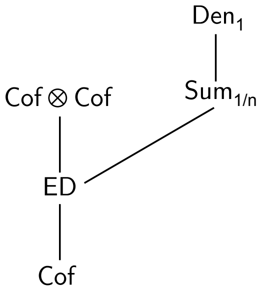
- given an instance $x \in \mathbf{dom}(F)$,
- there exists a majority $A \in \mathcal{U}$,
- and then, given any member z of the majority A,
- a reduction ψ yields a solution $y \in F(x)$.

	Player I	*Player II*
1:	$x \in \mathbf{dom}(F)$	
2:		$A \in \mathcal{U}$
3:	$z \in A$	
4:		$y \in F(x)$

▷ For any instance x, there is a *majority* $A \in \mathcal{U}$ such that $\psi(x, z)$ outputs correct solutions for $F(x)$ on any $z \in A$.

Examples. The following are filters on ω.

- Cof is the set of all cofinite subsets of ω.
- Sum_f is the set of all $A \subseteq \omega$ such that $\sum_{n \notin A} f(n) < \infty$.
- Den_1 is the set of all $A \subseteq \omega$ whose asymptotic density is one; that is, $\lim_{n\to\infty} |A \cap n|/n = 1$.
- ED is the set of all $A \subseteq \omega^2$ such that $\{n : |A^{[n]}| \leq m\} \in Cof$ for some m, where $A^{[n]} = \{k : (n,k) \in A\}$.



How to make many queries?

- *Compositional product $\star$: $F \leq_W G \star H$:*
  - The computation first makes a query to H once, and then makes a query to G once.

$F \leq_W G \star H \iff$
Player II has a computable winning strategy for the following game:

	Player I	*Player II*
1:	$x_0 \in \mathrm{dom}(F)$	
2:		$z_0 \in \mathrm{dom}(H)$
3:	$x_1 \in H(z_0)$	
4:		$z_1 \in \mathrm{dom}(G)$
5:	$x_2 \in G(z_1)$	
6:		$z_2 \in F(x_0)$

- F is n-queries reducible to $G \iff F \leq_W \underbrace{G \star \cdots \star G}_{n}$

A majority version?

- *Fubini product* $\otimes$: $\mathcal{U} \leq_K \mathcal{V} \otimes \mathcal{W}$:
  - ▷ $A \in \mathcal{U} \otimes \mathcal{V} \iff \{n \in \omega : A^{[n]} \in \mathcal{V}\} \in \mathcal{U}$.

$\mathcal{U} \leq_K \mathcal{V} \otimes \mathcal{W} \iff$
Player II has a winning strategy for the following game:

	Player I	Player II
1:	$A \in \mathcal{U}$	
2:		$B_0 \in \mathcal{V}$
3:	$x_0 \in B_0$	
4:		$B_1 \in \mathcal{W}$
5:	$x_1 \in B_1$	
6:		$y \in A$

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... "*oracle*" and "*majority*" in logic.

- *Idea:* an *oracle* is a transcendent axiom?
- *Idea:* a *majority notion* relaxes provability? It is sufficient for a majority to have succeeded in proving a statement.
- Anyway, both are factors that strengthen a theory.

Want to compare the strengths of various theories.

- ▹ Computability Theory < Set Theory < Inconsistent Theory
- ▹ Oracle and majority seem likely to be useful for analyzing theories at intermediate levels.

... "*oracle*" and "*majority*" play key roles in effective topos theory.

- Hyland (1981) introduced the *effective topos*.
  - ▹ The world of *computable mathematics*.
- Want to know: The relationship with *other toposes*.
  - ▹ How it relate to different worlds of mathematics?

Subtoposes $\approx$ stronger theories.

- Idea: Stronger theory $\approx$ smaller Lindenbaum algebra.
  - ▷ Computability Theory < Set Theory < Inconsistent Theory
  - ▷ The eff. topos $\supset$ The topos of $\boldsymbol{Sets}$ $\supset$ The degenerate topos

What are *subtoposes* of the effective topos?

- ▷ The degenerate topos is the smallest subtopos of the eff. topos.
- ▷ $\boldsymbol{Sets}$ is the second smallest subtopos of the eff. topos.
- ▷ Other subtoposes must correspond to

  intermediate worlds between $\boldsymbol{Eff}$ and $\boldsymbol{Sets}$

  — or intermediate theories between Computability Theory and Set Theory.

An *oracle* always yields a *subtopos*.

- Hyland (1981): Turing degrees (single-valued oracles) embed into the subtoposes of the effective topos.
  - $\mathit{Eff}[A]$: the world of A-relative computable mathematics.
  - $A \leq_{\mathsf{T}} B \iff \mathit{Eff}[B] \subseteq \mathit{Eff}[A]$.
- K. (2023): The $\mathbb{N}$-ver. of G.Weihrauch degrees (partial multi-valued oracles) embed into the subtoposes of the effective topos.
  - $G \leq_{\mathsf{GW}} F \iff \mathit{Eff}[F] \subseteq \mathit{Eff}[G]$.

($\subseteq$ means the existence of a geometric inclusion)

Hirschfeldt-Jockusch's reduction game (2016):

Player I	*Player II*
$x_0 \in \mathbf{dom}(F)$	
	Query: $z_0 \in \mathbf{dom}(G)$
$x_1 \in G(z_0)$	
	Query: $z_1 \in \mathbf{dom}(G)$
$x_2 \in G(z_1)$	
$\vdots$	$\vdots$
	Query: $z_{n-1} \in \mathbf{dom}(G)$
$x_n \in G(z_{n-1})$	
	Halt: $z_n \in F(x_0)$

- *Player I*'s moves are $x_0, x_1, x_2, \ldots$
- *Player II*'s moves are of the forms $(\mathtt{Query}, z_i)$ or $(\mathtt{Halt}, z_i)$.
  - `Query` is a signal to ask a query to oracle.
  - `Halt` is a signal to terminate the computation.
- F is *G.Weihrauch reducible* to G (written $F \leq_{\mathrm{GW}} G$)
  $\iff$ *Player II* has a computable winning strategy.

What about ... a *majority*?

- Pitts (1981) discovered a subtopos $\mathbf{Eff}[\mathbf{Cof}]$ that is different from any Turing subtopos $\mathbf{Eff}[A]$.
  - Intermediate: $\mathbf{Set} \subsetneq \mathbf{Eff}[\mathbf{Cof}] \subsetneq \mathbf{Eff}$.
  - No bound by a Turing oracle: $\mathbf{Eff}[A] \not\subseteq \mathbf{Eff}[\mathbf{Cof}]$ for any $A \subseteq \omega$

Pitts' subtopos is ... a *"majority"*-based computability!

- Run ω many computations simultaneously, and
  if all but a finite number of them are successful, then it is OK.
- In other words, it is computability w.r.t. the *cofinite filter* $\mathbf{Cof}$.

A majority notion always yields a *subtopos*!

- This is our new perspective (K.-Ng), not Hyland-Pitts' perspective. Reaching this perspective was a kind of breakthrough.
  - Probably, the importance of *majority* (filter) in the study of subtopos was not recognized for a long time.
  - For more than 40 years since Hyland and Pitts, no examples of filter-based subtopos were discovered except for $\mathit{Eff}[\mathit{Cof}]$.

Lee-van Oosten (2011,13) came closest to the *"majority"* idea. What they dealt with, from our perspective, is:

- Run n many computations simultaneously, and if k of them are successful, then it is OK.

However, since what they dealt with was a set of subsets of $\mathbb{N}$ rather than a filter on $\mathbb{N}$, the relevance to (ultra-)filter theory was not considered.

Okay, so both *oracle* and *majority* yield subtoposes. Is that all?

- Lee-van Oosten (2011,13) provided a method for presenting all subtoposes of the effective topos.
- K. (2023) explained that Lee-van Oosten's presentation consists of two layers, *bilayer function*.

So, the answer to the above question is: Yes, that's all! Nothing else.

- ▷ Oracle + Majority ≈ subtoposes of $\mathcal{E}\!f\!f$.

Other related work:

- K. (2023) also pointed out that Bauer (2022)'s *extended Weihrauch predicate* is the same as a bilayer function.

"Oracle" and *"majority"*: Bilayer view of extended Weihrauch predicate.

- The Weihrauch part is the oracle layer.
- The extended part is the majority layer.

Summary: A subtopos is presented by a bilayer of oracle/majority.

- The Oracle layer:
  - ▷ (Hyland 1981) Computability with Turing oracle.
  - ▷ (K. 2023) Computability with G.Weihrauch oracle.
- The Majority layer:
  - ▷ (Pitts 1981) Computability with the cofinite filter.
  - ▷ (Lee-van Oosten 2011,13) Computability with k/n success.
  - ▷ (K. 2023) Computability with the density one filter.

Review of past research:

- The Majority layer only has results for a few individual examples, and no unified theory existed in the past.
- Recent studies of extended Weihrauch degrees almost ignore the Majority layer (or are unaware of its importance).

Our claim (K.-Ng):

- The Majority layer is important and is linked to a very rich field of research (i.e., Rudin-Keisler order, Katětov order, etc.)

(K. 2023) The Oracle layer is completely determined by G.Weirauch reducibility.

- ▷ G.Weihrauch reducibility
  = Weihrauch reducibility + iterated compositional products

Recall:

- Weihrauch reducibility ≈ Katětov reducibility.
- compositional product ≈ Fubini product.

(K.-Ng) The Majority layer is completely determined by computable G.Katětov reducibility.

- ▷ G.Katětov reducibility
  = Katětov reducibility + iterated Fubini products

Hirschfeldt-Jockusch's reduction game (2016):

Player I	***Player II***
$x_0 \in \mathbf{dom}(F)$	
	Query: $z_0 \in \mathbf{dom}(G)$
$x_1 \in G(z_0)$	
	Query: $z_1 \in \mathbf{dom}(G)$
$x_2 \in G(z_1)$	
$\vdots$	$\vdots$
	Query: $z_{n-1} \in \mathbf{dom}(G)$
$x_n \in G(z_{n-1})$	
	Halt: $z_n \in F(x_0)$

- *Player I*'s moves are $x_0, x_1, x_2, \ldots$
- *Player II*'s moves are of the forms $(\texttt{Query}, z_i)$ or $(\texttt{Halt}, z_i)$.
  - `Query` is a signal to ask a query to oracle.
  - `Halt` is a signal to terminate the computation.
- F is *G.Weihrauch reducible* to G (written $F \leq_{\mathrm{GW}} G$)
  $\iff$ *Player II* has a computable winning strategy.

Katětov game (K.-Ng):

Player I	*Player II*
$A \in \mathcal{U}$	
	Query: $B_0 \in \mathcal{V}$
$x_1 \in B_0$	
	Query: $B_1 \in \mathcal{V}$
$x_2 \in B_1$	
$\vdots$	$\vdots$
	Query: $B_{n-1} \in \mathcal{V}$
$x_n \in B_{n-1}$	
	Halt: $y \in A$

- *Player I*'s moves are $A, x_1, x_2, \dots$
- *Player II*'s moves are of the forms $(\mathtt{Query}, B_i)$ or $(\mathtt{Halt}, y)$.
- F is *(computable) G.Katětov reducible* to G
  $\iff$ *Player II* has a (computable) winning strategy.
  - Here, we consider computability for $x_1, \dots, x_n \mapsto \mathtt{Query}$ or $(\mathtt{Halt}, y)$.

Oracle layer:

Theorem (K. 2023)

The following are equivalent for partial multifunctions $F, G :\subseteq \mathbb{N} \rightrightarrows \mathbb{N}$:

1. *F is G.Weihrauch reducible to G.*
2. *There is a geometric inclusion $\mathit{Eff}[G] \hookrightarrow \mathit{Eff}[F]$.*

Majority layer:

Theorem (K.-Ng)

The following are equivalent for upper sets $\mathcal{U}, \mathcal{V} \subseteq \mathcal{P}(\mathbb{N})$:

1. *$\mathcal{U}$ is computable G.Katětov reducible to $\mathcal{V}$.*
2. *There is a geometric inclusion $\mathit{Eff}[\mathcal{V}] \hookrightarrow \mathit{Eff}[\mathcal{U}]$.*

Merging the Oracle and Majority layers:

Def. A sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of upper sets $\mathcal{U}_n \subseteq \mathcal{P}(\mathbb{N})$ is called an upper sequence.

- An upper set $\mathcal{U}$ is identified with a *constant* upper sequence.
- A single-valued function $f : \mathbb{N} \to \mathbb{N}$ is identified with a *principal-ultrafilter*-valued upper sequence.
  - A single-value $\{n\}$ generates a principal ultrafilter.
- A multifunction $f : \mathbb{N} \rightrightarrows \mathbb{N}$ is identified with a *principal-filter*-valued upper sequence.
  - A multi-value $A \subseteq \mathbb{N}$ generates a principal filter.

- Extended G.Weihrauch reducibility (a.k.a. LT-reducibility) for upper sequences extends both G.Weihrauch reducibility and computable G.Katětov reducibility.
- This completely characterizes the subtoposes of the effective topos.

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- Sum_f is the set of all $A \subseteq \mathbb{N}$ such that $\sum_{n \in A} f(n) < \infty$.
- A *summable ideal* is an ideal of the form Sum_f for some $f : \mathbb{N} \to \mathbb{N}$.

Theorem (K.-Ng)

There is an order embedding of $(\mathcal{P}(\mathbb{N}), \subseteq^*)$ into the computable G.Katětov order on (the duals of) summable ideals.

This shows that the structure of *"majority-based"* subtoposes of Eff is extremely vast.

- There is an uncountable chain of majority-based subtoposes of Eff.
- There is a size $2^{\aleph_0}$ antichain of majority-based subtoposes of Eff.

The Majority layer shouldn't be ignored!

The power of "computability by majority."

- Pitts (1981) found $\mathit{Eff}[\mathit{Cof}]$.
- Van Oosten (2014): If f is single-valued,

$$f \leq_{\mathrm{LT}} \mathit{Cof} \iff f \text{ is hyperarithmetic.}$$

  - Van Dijk (2017): One can conclude that $\mathit{Eff}[\mathit{Cof}]$ is 'the world of hyperarithmetical mathematics.'
  - Is it really "*the*?"
- K. (2023): If f is single-valued,

$$f \leq_{\mathrm{LT}} \mathit{Den}_1 \iff f \text{ is hyperarithmetic.}$$

  - $\mathit{Eff}[\mathit{Den}_1]$ is also a world of hyperarithmetical mathematics.

The power of "computability with majority."

Theorem (K.-Ng)

Let $\mathcal{U}$ be a non-principal Δ^1_1 filter on $\mathbb{N}$. For any $f\colon \mathbb{N} \to \mathbb{N}$,

$$f \leq_{\mathrm{LT}} \mathcal{U} \iff f \text{ is hyperarithmetic.}$$

The subtopos obtained from a natural filter $\mathcal{U}$ is almost always a "world of hyperarithmetical mathematics!"

Summary:

- "*Computability by Majority*" is a rich research field!
  - ▷ Compared by computable Katětov order.
- Surprisingly, this is also related to the structure of the *subtoposes of the effective topos*.
- This Majority layer is also the "extended" part of extended Weihrauch reducibility.
  - ▷ This "extended" part is currently almost ignored as a minor part, but it is by no means a minor part...

Thank you for your attention!