Randomness and the Enumeration Degrees

Joseph S. Miller
University of Wisconsin–Madison

CCR 2025: Computability, Complexity, and Randomness
16–20 June 2025, Bordeaux

Outline

- Part 1: A quick intro to the enumeration degrees.
- Part 2: Quasiminimality and measure / randomness.
- Part 3: Randomness relative to an enumeration oracle.

Part 1: A quick introduction to the enumeration degrees

The enumeration degrees

Definition (Rogers 1957). $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leq_e B$) if there is a uniform procedure to enumerate A from any enumeration of B.

Here, an *enumeration* of $B \subseteq \omega$ is any $f \in \omega^\omega$ with $\mathrm{range}(f) = B$. (Fine, add a padding symbol for the empty set.)

Thm. (Selman 1971). Equivalently, we can drop the uniformity.

Sticking with uniformity, the definition is usually written as:

Definition. $A \leq_e B$ if there is a c.e. set Γ such that

$$A = \{n \colon (\exists e)\ \langle n, e\rangle \in \Gamma \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set. We call Γ an *enumeration operator*.

The induced degree structure $\mathcal{D}_e$ is the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

The total enumeration degrees

Proposition. $X \leq_T Y$ iff $X \oplus \overline{X}$ is Y-c.e. iff $X \oplus \overline{X} \leq_e Y \oplus \overline{Y}$.

This suggests a natural embedding of the Turing degrees into the enumeration degrees.

Proposition. The embedding $\iota \colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

$$\iota(d_T(X)) = d_e(X \oplus \overline{X}),$$

preserves the order and the least upper bound.

Definition. The *total degrees* are the image of the Turing degrees under this embedding (i.e., they are the enumeration degrees that contain a set of the form $X \oplus \overline{X}$).

- An e-degree is total iff it contains the graph a total function.
- The degrees of (graphs of) enumerations of A are exactly the total degrees above A. By Selman's theorem, these determine the e-degree of A.

Quasiminimality

The first construction of a nontotal enumeration degree had an interesting property.

Definition. A non-c.e. set $A \subseteq \omega$ has *quasiminimal* enumeration degree if $X \oplus \overline{X} \leq_e A$ implies that X is computable.

I.e., $\deg_e(A) \neq \mathbf{0}_e$ and the only total degree below $\deg_e(A)$ is $\mathbf{0}_e$.

- (Medvedev 1955) There is a quasiminimal enumeration degree.
- (Myhill 1961) The sets with quasiminimal e-degree are comeager. (Copestake 1988) 1-generic sets have quasiminimal e-degree.
- (Lagemann 1971) Almost every set has quasiminimal e-degree.
- (McEvoy 1985) Every total degree above $\mathbf{0}_e'$ is the jump of a quasiminimal e-degree.
- (Arslanov, Cooper, and Kalimullin 2003) Semicomputable sets that are not c.e. or co-c.e. have quasiminimal e-degree.

Part 2: Quasiminimality and measure/randomness

(with Cholak and Soskova)

Lagemann's results

In his dissertation, Lagemann gave the first results involving measure and the enumeration degrees (verifying conjectures of John Case).

> *"I wrote my dissertation on a hopelessly abstruse topic in mathematical logic..." – Jay Lagemann*

Theorem (Lagemann 1971)

(1) $A \mid_e \overline{A}$ for almost every $A \subseteq \omega$.

(2) Almost every A is quasiminimal.

- Note that the first result follows easily from the second.
- We will sketch a proof of (2).
- Then we will show that every weakly 2-random is quasiminimal.

A little probability / combinatorics

At a key step, Lagemann claims that, for a fixed enumeration operator Γ and a random $A \subseteq \omega$,

$$\Pr(m \in \Gamma^A \mid n \in \Gamma^A) \geq \Pr(m \in \Gamma^A).$$

He says that this "inequality holds because [Γ] is an enumeration operator. The condition [$n \in \Gamma^A$] only forces A to contain more members which by the monotonicity of enumeration operators makes it more likely that [$m \in \Gamma^A$]."

We capture this idea using "upsets": $M \subseteq 2^k$ is an *upset* if whenever $\sigma \in M$ and $\sigma \subseteq \tau \in 2^k$, then $\tau \in M$. (Forgive the abuse of notation.)

Theorem (Harris 1960; Kleitman 1966). If $M, N \subseteq 2^k$ are upsets, then

$$\frac{|M \cap N|}{2^k} \geq \frac{|M|}{2^k} \cdot \frac{|N|}{2^k}.$$

A little probability / combinatorics

Theorem (Harris 1960; Kleitman 1966). If $M, N \subseteq 2^k$ are upsets, then

$$\frac{|M \cap N|}{2^k} \geq \frac{|M|}{2^k} \cdot \frac{|N|}{2^k}.$$

The Harris–Kleitman inequality states that upsets are non-negatively correlated. In other words,

$$\Pr(\sigma \in M \mid \sigma \in N) \geq \Pr(\sigma \in M).$$

Extending the notion of *upset* to subsets of 2^ω (i.e., subsets of $\mathcal{P}(\omega)$):

The Harris–Kleitman inequality on 2^ω. If $M, N \subseteq 2^\omega$ are measurable upsets, then

$$\mu(M \cap N) \geq \mu(M)\mu(N).$$

Almost every set has quasiminimal e-degree

Theorem (Lagemann 1971). Almost every A is quasiminimal.

Proof. Fix an enumeration operator Γ and assume, for a contradiction, that

$$P = \{A \in 2^\omega \colon (\exists X)\ X \text{ is not computable and } \Gamma^A = X \oplus \overline{X}\}$$

has positive measure.

- By Lebesgue density, there is a $\sigma \in 2^{<\omega}$ such that the relative measure of P above σ is as large as we like, say
$$\frac{\mu(P \cap [\sigma])}{\mu([\sigma])} > \frac{6}{7}.$$
- We could "hard-code" σ into Γ, so without loss of generality, assume that $\mu(P) > 6/7$.
- **Goal:** find a partition $P = P_0 \sqcup P_1$ such that $\mu(P_0), \mu(P_1) > 3/7$ and if $A_0 \in P_0$ and $A_1 \in P_1$, then $\Gamma^{A_0} \neq \Gamma^{A_1}$.

Almost every set has quasiminimal e-degree

Proof (cont.).

- Consider the measure λ such that, for each Borel $M \subseteq 2^\omega$,
$$\lambda(M) = \mu\{A \in P \colon (\exists X \in M)\ \Gamma^A = X \oplus \overline{X}\}.$$
- Essentially, λ is the push-forward of $\mu\restriction_P$ under Γ. In particular, $\lambda(2^\omega) = \mu(P) > 6/7$.
- Note that λ is atomless because if X is noncomputable, then
$$\mu\{A \in 2^\omega \colon X \oplus \overline{X} \leq_e A\} \leq \mu\{A \in 2^\omega \colon X \leq_T A\} = 0$$
(De Leeuw, Moore, Shannon, and Shapiro 1956; Sacks 1963).
- Since λ is atomless, it is as small as we like on all neighborhoods in a sufficiently fine clopen partition of 2^ω.
- Take M_0, M_1 to be disjoint clopen sets with $\lambda(M_i) > 3/7$ for each $i \in \{0, 1\}$.

Almost every set has quasiminimal e-degree

Proof (cont.).

- Take M_0, M_1 to be disjoint clopen sets with $\lambda(M_i) > 3/7$ for each $i \in \{0,1\}$.

- Let $P_i = \{A \in P \colon (\exists X \in M_i)\ \Gamma^A = X \oplus \overline{X}\}$ for each $i \in \{0,1\}$. Note that $\mu(P_i) = \lambda(M_i) > 3/7$.

- If $A_0 \in P_0$ and $A_1 \in P_1$, then $\Gamma^{A_0} \neq \Gamma^{A_1}$. Thus, if $A_0, A_1 \subseteq C$, then Γ^C does not have the form $X \oplus \overline{X}$, and so $C \notin P$.

- Let $\check{P}_i$ be the upset generated by P_i for each $i \in \{0,1\}$. By above, $\check{P}_0 \cap \check{P}_1$ is disjoint from P.

- Using the Harris–Kleitman inequality,

$$1/7 > \mu(\check{P}_0 \cap \check{P}_1) \geq \mu(\check{P}_0)\mu(\check{P}_1) \geq \mu(P_0)\mu(P_1) > (3/7)^2 = 9/49,$$

  which is a contradiction. □

Every weakly 2-random has quasiminimal e-degree

For an enumeration operator Γ, consider the class

$$Q = \{A \in 2^\omega \colon (\exists X)\, \Gamma^A = X \oplus \overline{X} \text{ and } A \text{ is not } X\text{-random}\}.$$

Claim. $\mu(Q) = 0$.

Proof. Almost all A are 1-random *and* quasiminimal. Such an A is not in Q because it cannot be derandomized by a computable X. □

Claim. Q is a Π^0_2 class.

Proof. Let $\{U_n^{\square}\}_{n\in\omega}$ be a universal oracle Martin-Löf test. Then, $A \in Q$ if and only if

- $(\forall n)(\forall s)\, \neg(2n, 2n+1 \in \Gamma^A_s)$,
- $(\forall n)(\exists s)\, 2n \in \Gamma^A_s$ or $2n+1 \in \Gamma^A_s$, and
- $(\forall n)(\exists s)(\exists \sigma \in 2^{<\omega})\, \sigma \oplus \overline{\sigma} \subseteq \Gamma^A_s$ and $A \in U^\sigma_{n,s}$. □

Every weakly 2-random has quasiminimal e-degree

For an enumeration operator Γ, consider the class

$$Q = \{A \in 2^\omega \colon (\exists X)\, \Gamma^A = X \oplus \overline{X} \text{ and } A \text{ is not } X\text{-random}\}.$$

Claim. Q is a measure zero Π^0_2 class.

Theorem (Cholak, M., Soskova). If A is weakly 2-random, then it has quasiminimal e-degree.

Proof. Assume that $\Gamma^A = X \oplus \overline{X}$ for some enumeration operator Γ.

- Note that $A \notin Q$, so A must be random relative to X.
- But A computes X, so X is a base for 1-randomness (hence K-trivial). This implies that X is Δ^0_2.
- However, weak 2-random sets cannot compute noncomputable Δ^0_2 sets, so X must be computable. □

Collecting our thoughts

The use of K-triviality is a little surprising. Is it avoidable?

Question. Is there a more "natural" proof that weakly 2-random sets have quasiminimal enumeration degree?

Fact. Not every 1-random set has quasiminimal enumeration degree.

- $\overline{\Omega} \geq_e \Omega$. Therefore, $\overline{\Omega} \equiv_e \emptyset' \oplus \overline{\emptyset'}$, so it has total degree.
- Similarly, $\overline{\Omega \oplus \Omega^{\emptyset'}}$ has total degree equivalent to $\emptyset'' \oplus \overline{\emptyset''}$.
- Using a result of Hirschfeldt, Jockusch, Kuyper, and Schupp, "Coarse reducibility and algorithmic randomness," 2016, we show:

Corollary. If $X \leq_T \emptyset'$ is 1-random, then it does not have quasiminimal enumeration degree.

Question. Is there a 1-random that has quasiminimal e-degree but is not weakly 2-random?

The partial computable extension property

Kalimullin and Puzarenko (2004) introduced a property stronger than quasiminimality (prefigured by Copestake 1988). Let $G_\psi \subseteq \omega^2$ denote the graph of a partial function $\psi\colon \omega \to \omega$.

Definition. $A \subseteq \omega$ has the *partial computable extension property (PCEP)* if whenever $G_\psi \leq_e A$, there is a partial computable $\varphi \supseteq \psi$.

- $X \oplus \overline{X} \equiv_e G_{\mathbf{1}_X}$, where $\mathbf{1}_X$ is the characteristic function of X.

$\Rightarrow$ If A has the PCEP and $X \oplus \overline{X} \leq_e A$, then X is computable.

$\Rightarrow$ If A has the PCEP, it has quasiminimal degree (or is c.e.).

- Simple constructions of quasiminimal sets usually yield the PCEP, e.g., 1-generic and nontrivial semicomputable sets.

  But not graphs of 1-generic partial functions (Copestake 1988).

- (Cholak, M., Soskova) No 1-random has the PCEP.

Part 3: Randomness relative to an enumeration oracle

(with Greenberg and Soskova)

Using an enumeration oracle

- Most fundamental notions in effective randomness can be expressed in terms of c.e. sets.
- E.g., Σ^0_1 class, Martin-Löf test, Solovay test, left-c.e. semimeasure, Kolmogorov complexity, ...
- We relativize these notions to a Turing oracle X using X-c.e. sets (not X-computable sets).
- Main Idea. To relativize to an enumeration oracle A, simply replace "X-c.e." with "$\leq_e A$".

Definition. Consider a set of strings $W \subseteq 2^{<\omega}$ such that $W \leq_e A$. We call $[W] = \{X \in 2^\omega : (\exists \sigma \in W)\ \sigma \prec X\}$ a *$\Sigma^0_1\langle A\rangle$ class*.

- We use $\langle A\rangle$ to indicate that A is an enumeration oracle.

Using an enumeration oracle

- We use $\langle A \rangle$ to indicate that A is an enumeration oracle.
- Note: a $\Sigma^0_1\langle X \oplus \overline{X} \rangle$ class is just a $\Sigma^0_1(X)$ class in the usual sense.
- The complement of a $\Sigma^0_1\langle A \rangle$ class is a $\Pi^0_1\langle A \rangle$ *class*.

Aside. These notions have proved useful. For example:

The *continuous degrees* are the degrees of points in computable metric spaces. They properly extend the Turing degrees and properly embed into the enumeration degrees (M. 2004).

Theorem (Andrews, Igusa, M., Soskova 2019). TFAE:

(1) $A \subseteq \omega$ has continuous enumeration degree,

(2) There is a nonempty $\Pi^0_1\langle A \rangle$ class P such that A is (uniformly) X-c.e. for every $X \in P$, and

(3) A has *almost total* enumeration degree: whenever $\mathbf{b} \not\leq \deg_e(A)$ is total, $\deg_e(A) \vee \mathbf{b}$ is also total.

Defining $\langle A \rangle$-randomness

Definition. A *Kučera $\langle A \rangle$-test* is a $\Sigma^0_1\langle A \rangle$ class U of measure less than one. We say that $Z \in 2^\omega$ *fails* this test if every tail of Z is in U. We call Z $\langle A \rangle$-random if it fails no Kučera $\langle A \rangle$-tests.

We can also define *Martin-Löf $\langle A \rangle$-tests* and *$\langle A \rangle$-enumerable (super)martingales*. The resulting randomness notions are equivalent to $\langle A \rangle$-randomness.

Not everything works so well. Depending on A,

- Solovay $\langle A \rangle$-tests might give a weaker notion, at least if viewed as sets of strings $\leq_e A$ of finite weight.
- So might prefix-free complexity relative to $\langle A \rangle$ (even if defined in terms of "request sets").
- There might not be a universal $\langle A \rangle$-test.

Of course, things work as expected for total enumeration degrees: $\langle X \oplus \overline{X} \rangle$-randomness is just X-randomness.

Why do things go wrong?

Given a fixed enumeration of A, we can

(1) Uniformly enumerate a prefix-free set of strings that generate the eth $\Sigma^0_1\langle A\rangle$ class. (Prefix-free sets are used in many basic proofs.)

(2) Uniformly enumerate strings that generate the eth $\Sigma^0_1\langle A\rangle$ class, but stop if the class is about to violate a given measure bound. (Restricting measure is used to build universal tests.)

However, *different* enumerations of A would usually result in *different* prefix-free sets in (1), or different subclasses in (2).

- It is possible to have a $\Sigma^0_1\langle A\rangle$ class that is not generated by any prefix-free set of strings $\leq_e A$.
- Sometimes we can find a different proof, e.g., to prove that every Kučera $\langle A\rangle$-test is covered by a Martin-Löf $\langle A\rangle$-test.
- Other times. . . well, this talk will not focus on separations between $\langle A\rangle$-randomness notions.

What *is* our focus?

- We want to compare $\langle A \rangle$-randomness to randomness relative to the total degrees above and below A.
- In part, to show that $\langle A \rangle$-randomness is not easily captured by randomness relative to Turing oracles.
- (Recall that $\deg_e(A)$ is uniquely determined by the total degrees above A.)
- It will be important whether or not there is a universal $\langle A \rangle$-test.

Proposition. There is a universal Kučera $\langle A \rangle$-test

$\iff$ there is a universal Martin-Löf $\langle A \rangle$-test

$\iff$ there is a *universal* $\langle A \rangle$-enumerable (super)martingale

$\iff$ there is a positive measure $\Pi^0_1\langle A \rangle$ class containing only $\langle A \rangle$-randoms.

Interaction with total oracles

Theorem (Greenberg, M., Soskova). For a set $A \subseteq \omega$ and sequence $Z \in 2^\omega$, consider the following notions:

(I) Z is X-random for some X such that $A \leq_e X \oplus \overline{X}$,

(II) Z is $\langle A \rangle$-random,

(III) Z is Y-random for every Y such that $Y \oplus \overline{Y} \leq_e A$.

Then (I) $\Rightarrow$ (II) $\Rightarrow$ (III). Furthermore, each implications can be strict.

- The implications are easy: if $A \leq_e B$, then $\langle B \rangle$-randomness implies $\langle A \rangle$-randomness.
- It is not hard to show that A is never $\langle A \rangle$-random.
- If A is weakly 2-random, then A is random relative to every total degree below A (it is quasiminimal!), but not $\langle A \rangle$-random. Therefore, (III) $\not\Rightarrow$ (II) for A.

Lowness for randomness

I owe you an example for which (II) $\not\Rightarrow$ (I), but first:

Definition. For a set A, we say that *$\langle A \rangle$ is low for randomness* if every 1-random is $\langle A \rangle$-random.

- In this case, (III) $\Rightarrow$ (II) for A.
- 1-generic and nontrivial semicomputable sets are low for randomness in the enumeration degrees.
- It should not be surprising that these examples are quasiminimal. Lowness for randomness is closed downward and only countably many total degrees are low for randomness: $\deg_e(X \oplus \overline{X})$ where X is K-trivial.
- We know that not every quasiminimal e-degree is low for randomness: weakly 2-randoms.
- Good question! But no, not every A with the partial computable extension property is low for randomness.

A couple of examples

Lemma. Assume that A is weakly 1-generic relative to Z. If A is X-c.e., then Z is not X-random.

Example. Let A be weakly 2-generic.

- A is low for randomness.
- A is weakly 1-generic relative to Ω. So, by the lemma, if $A \leq_e X \oplus \overline{X}$, then Ω is not X-random.
- Therefore, (II) $\not\Rightarrow$ (I) for A.

When does (III) $\Rightarrow$ (I) for A? I.e., when are all three equivalent? This is true for total degrees, vacuously, but also for some nontotal degrees.

Example. Let X be a noncomputable K-trivial.

- Let $A \leq_e X \oplus \overline{X}$ be nontotal. (Jockusch (1968) proved that if $X \not\leq_T \emptyset$, then there is a nontrivial semicomputable set $A \equiv_T X$.)
- Every 1-random is X-random, so (I) $\Leftrightarrow$ (II) $\Leftrightarrow$ (III) for A.

Trivial and nontrivial implications

The previous example feels a little too "trivial". Recall:

(I) Z is X-random for some X such that $A \leq_e X \oplus \overline{X}$,

(II) Z is $\langle A \rangle$-random,

(III) Z is Y-random for every Y such that $Y \oplus \overline{Y} \leq_e A$.

Definition. For $A \subseteq \omega$, we say that

- (II) $\Rightarrow$ (I) *trivially for A* if there is an X such that $A \leq_e X \oplus \overline{X}$ and $\langle X \oplus \overline{X} \rangle$ is low for randomness with respect to $\langle A \rangle$.
- (III) $\Rightarrow$ (II) *trivially for A* if there is a Y such that $Y \oplus \overline{Y} \leq_e A$ and $\langle A \rangle$ is low for randomness with respect to $\langle Y \oplus \overline{Y} \rangle$.

We will see that each of (III) $\Rightarrow$ (II) and (II) $\Rightarrow$ (I) can hold *nontrivially* for A.

- But not both at the same time!
- And it matters whether or not there is a universal $\langle A \rangle$-test.

How does it all relate?

		(II) $\not\Rightarrow$ (I)	(II) $\Rightarrow$ (I) trivially	(II) $\Rightarrow$ (I) nontrivially
Universal $\langle A \rangle$-test	(III) $\not\Rightarrow$ (II)	✓	✓	✓
	(III) $\Rightarrow$ (II) trivially	✓	✓	?
	(III) $\Rightarrow$ (II) nontrivially	✗		
No universal $\langle A \rangle$-test	(III) $\not\Rightarrow$ (II)	✓	✗	
	(III) $\Rightarrow$ (II) trivially	✗		
	(III) $\Rightarrow$ (II) nontrivially	✓		

The interaction of $\langle A \rangle$-randomness, the total degrees above and below the degree of A, and whether or not there is a universal $\langle A \rangle$-test.

Generics and randoms

<table>
<tr><th></th><th></th><th>(II) $\not\Rightarrow$ (I)</th><th>(II) $\Rightarrow$ (I) trivially</th><th>(II) $\Rightarrow$ (I) nontrivially</th></tr>
<tr><td rowspan="3">Universal $\langle A \rangle$-test</td><td>(III) $\not\Rightarrow$ (II)</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>(III) $\Rightarrow$ (II) trivially</td><td>✓</td><td>✓</td><td>?</td></tr>
<tr><td>(III) $\Rightarrow$ (II) nontrivially</td><td colspan="3">✗</td></tr>
<tr><td rowspan="3">No universal $\langle A \rangle$-test</td><td>(III) $\not\Rightarrow$ (II)</td><td>✓</td><td colspan="2" rowspan="3">✗</td></tr>
<tr><td>(III) $\Rightarrow$ (II) trivially</td><td>✗</td></tr>
<tr><td>(III) $\Rightarrow$ (II) nontrivially</td><td>✓</td></tr>
</table>

- Weakly 2-generics are here.
- I don't know where sufficiently random sets belong.

The total degrees and our “trivial” example

<table>
<tr><th></th><th></th><th>(II) $\not\Rightarrow$ (I)</th><th>(II) $\Rightarrow$ (I) trivially</th><th>(II) $\Rightarrow$ (I) nontrivially</th></tr>
<tr><td rowspan="3">Universal $\langle A \rangle$-test</td><td>(III) $\not\Rightarrow$ (II)</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>(III) $\Rightarrow$ (II) trivially</td><td>✓</td><td>✓</td><td>?</td></tr>
<tr><td>(III) $\Rightarrow$ (II) nontrivially</td><td colspan="3">✗</td></tr>
<tr><td rowspan="3">No universal $\langle A \rangle$-test</td><td>(III) $\not\Rightarrow$ (II)</td><td>✓</td><td colspan="2" rowspan="3">✗</td></tr>
<tr><td>(III) $\Rightarrow$ (II) trivially</td><td>✗</td></tr>
<tr><td>(III) $\Rightarrow$ (II) nontrivially</td><td>✓</td></tr>
</table>

- A set A is here iff there are X and Y s.t. $Y \oplus \overline{Y} \leq_e A \leq_e X \oplus \overline{X}$ and $\langle X \oplus \overline{X} \rangle$ is low for randomness with respect to $\langle Y \oplus \overline{Y} \rangle$.

When does (III) $\Rightarrow$ (I) for A?

<table>
<tr><th></th><th></th><th>(II) $\not\Rightarrow$ (I)</th><th>(II) $\Rightarrow$ (I) trivially</th><th>(II) $\Rightarrow$ (I) nontrivially</th></tr>
<tr><td rowspan="3">Universal $\langle A \rangle$-test</td><td>(III) $\not\Rightarrow$ (II)</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>(III) $\Rightarrow$ (II) trivially</td><td>✓</td><td>✓</td><td>?</td></tr>
<tr><td>(III) $\Rightarrow$ (II) nontrivially</td><td colspan="3">✗</td></tr>
<tr><td rowspan="3">No universal $\langle A \rangle$-test</td><td>(III) $\not\Rightarrow$ (II)</td><td>✓</td><td colspan="2" rowspan="3">✗</td></tr>
<tr><td>(III) $\Rightarrow$ (II) trivially</td><td>✗</td></tr>
<tr><td>(III) $\Rightarrow$ (II) nontrivially</td><td>✓</td></tr>
</table>

- Note that this is only possible when (III) $\Rightarrow$ (II) trivially for A.

Question. Does it also imply that (II) $\Rightarrow$ (I) trivially for A?

When does (III) $\Rightarrow$ (I) for A?

As we just noted, (III) $\Rightarrow$ (II) trivially for A: there is a Y such that $Y \oplus \overline{Y} \leq_e A$ and $\langle A \rangle$ is low for randomness with respect to $\langle Y \oplus \overline{Y} \rangle$.

We might as well focus on the case where $Y = \emptyset$.

Definition. We say that $\langle A \rangle$ is *strongly low for randomness* if every 1-random is random relative to some total degree above A.

Our "trivial" example has this property. Is it the *only* example?

Question. If $\langle A \rangle$ is strongly low for randomness, must some total degree above A be low for randomness?

Let us rephrase the question, removing defined terms.

Question. If for every 1-random Z there is an X such that A is X-c.e. and Z is X-random, must A be c.e. in some K-trivial?

If A is a counterexample, then (II) $\Rightarrow$ (I) nontrivially for A.

When does (II) $\Rightarrow$ (I) nontrivially for A?

<table>
<tr><th></th><th></th><th>(II) $\not\Rightarrow$ (I)</th><th>(II) $\Rightarrow$ (I) trivially</th><th>(II) $\Rightarrow$ (I) nontrivially</th></tr>
<tr><td rowspan="3">Universal $\langle A\rangle$-test</td><td>(III) $\not\Rightarrow$ (II)</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>(III) $\Rightarrow$ (II) trivially</td><td>✓</td><td>✓</td><td>?</td></tr>
<tr><td>(III) $\Rightarrow$ (II) nontrivially</td><td colspan="3">✗</td></tr>
<tr><td rowspan="3">No universal $\langle A\rangle$-test</td><td>(III) $\not\Rightarrow$ (II)</td><td>✓</td><td colspan="2" rowspan="3">✗</td></tr>
<tr><td>(III) $\Rightarrow$ (II) trivially</td><td>✗</td></tr>
<tr><td>(III) $\Rightarrow$ (II) nontrivially</td><td>✓</td></tr>
</table>

- The only examples we have are the "diagonally not computably diagonalizable" continuous degrees.

The continuous enumeration degrees

Recall. The *continuous degrees* are the degrees of points in computable metric spaces. ... A has continuous e-degree $\iff$ there is a nonempty $\Pi^0_1\langle A\rangle$ class P such that A is X-c.e. for every $X \in P$.

(We only need $\Rightarrow$, the easy direction.)

Proposition. Assume that A has continuous enumeration degree. Then there is a universal $\langle A\rangle$-test and (II) $\Rightarrow$ (I) for A.

Proof. Let U^X be a universal Kučera X-test for every X.

- Then $V = \bigcap_{X\in P} U^X$ is a $\Sigma^0_1\langle A\rangle$ class (by compactness).
- But if $X \in P$, then $A \leq_e X \oplus \overline{X}$, so V contains all non-$\langle A\rangle$-randoms. It also has measure less than one.
- Therefore, V is a universal Kučera $\langle A\rangle$-test.
- Finally, if Z is $\langle A\rangle$-random, then it passes V. But then it passes U^X for some X, so it is X-random for some $X \in P$. □

The nontotal continuous e-degrees

		(II) $\not\Rightarrow$ (I)	(II) $\Rightarrow$ (I) trivially	(II) $\Rightarrow$ (I) nontrivially
Universal $\langle A \rangle$-test	(III) $\not\Rightarrow$ (II)	✓	✓	✓
	(III) $\Rightarrow$ (II) trivially	✓	✓	?
	(III) $\Rightarrow$ (II) nontrivially	✗		
No universal $\langle A \rangle$-test	(III) $\not\Rightarrow$ (II)	✓	✗	
	(III) $\Rightarrow$ (II) trivially	✗		
	(III) $\Rightarrow$ (II) nontrivially	✓		

- Every A with nontotal continuous e-degree is somewhere here.

But if $Y \oplus \overline{Y} \leq_e A$, then $\langle A \rangle$ bounds a member of each nonempty $\Pi^0_1(Y)$ class (M. 2004). So $\langle A \rangle$ is not low for randomness w.r.t. $Y \oplus \overline{Y}$.

Diagonally not computably diagonalizable (DNCD)

My original proof that nontotal continuous degrees exist produced a *DNCD* sequence. This is an $\alpha \in [0,1]^{\mathbb{N}}$ (the Hilbert cube) such that

$$(\forall e)\ \psi_e^\alpha \downarrow \implies \alpha(e) = \psi_e^\alpha,$$

where ψ_e is the eth partial computable function $[0,1]^{\mathbb{N}} \to [0,1]$.

- The continuous degrees embed naturally into the e-degrees.
- If $A \subseteq \omega$ "codes" α, then A has nontotal continuous e-degree.
- If $A \leq_e X \oplus \overline{X}$, then X computes a path in each nonempty $\Pi^0_1\langle A\rangle$ class.

  Idea. Given two $\Sigma^0_1\langle A\rangle$ predicates P_0 and P_1, consider ψ_e^α such that if only P_i holds, then $\psi_e^\alpha = 1 - i$.A total degree above A can uniformly find $i_e \in \{0,1\}$ such that $\alpha(e) \neq i_e$.
- Hence, $\langle X \oplus \overline{X}\rangle$ is not low for randomness with respect to $\langle A\rangle$.
- Therefore, (II) $\Rightarrow$ (I) nontrivially for A.

When does (II) $\Rightarrow$ (I) nontrivially for A?

Question. If (II) $\Rightarrow$ (I) nontrivially for A, must A have DNCD degree? Nontotal continuous degree?

If so, then every e-degree that is strongly low for randomness would be bounded by a total degrees that is low for randomness.

Question. Does (II) $\Rightarrow$ (I) nontrivially for every nontotal continuous enumeration degree?

The previous question is a variation of one I've had for a while.

Question. Does every nontotal continuous degree contain a DNCD sequence?

Aside. Bauer and Hanson recently used a DNCD sequence to construct a topos in which there are countably many (Dedekind) real numbers.

An example with no universal test

<table>
<tr><th></th><th></th><th>(II) $\not\Rightarrow$ (I)</th><th>(II) $\Rightarrow$ (I) trivially</th><th>(II) $\Rightarrow$ (I) nontrivially</th></tr>
<tr><td rowspan="3">Universal $\langle A \rangle$-test</td><td>(III) $\not\Rightarrow$ (II)</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>(III) $\Rightarrow$ (II) trivially</td><td>✓</td><td>✓</td><td>?</td></tr>
<tr><td>(III) $\Rightarrow$ (II) nontrivially</td><td colspan="3">✗</td></tr>
<tr><td rowspan="3">No universal $\langle A \rangle$-test</td><td>(III) $\not\Rightarrow$ (II)</td><td>✓</td><td colspan="2" rowspan="3">✗</td></tr>
<tr><td>(III) $\Rightarrow$ (II) trivially</td><td>✗</td></tr>
<tr><td>(III) $\Rightarrow$ (II) nontrivially</td><td>✓</td></tr>
</table>

- The $\langle$self$\rangle$-PA enumeration degrees are here. They are the easiest examples without universal tests.

$\langle$self$\rangle$-PA degrees

Definition. A set A is *$\langle$self$\rangle$-PA* if for every nonempty $\Pi^0_1\langle A\rangle$ class $P \subseteq 2^\omega$, there is an $Y \in P$ such that $Y \oplus \overline{Y} \leq_e A$.

- These are not hard to construct.
  - Fix sets $B_0 \ll B_1 \ll B_2 \ll \cdots$, each PA over the previous one.
  - On stage $2e$, copy B_e into the (tail of) the eth column of A.
  - On stage $2e+1$, add finitely much to A in columns $> e$ to make the eth $\Pi^0_1\langle A\rangle$ class empty, if possible.
  - If you can't, then it has a nonempty $\Pi^0_1\langle A^{[\leq e]}\rangle$ subclass.
  - But B_{e+1}, hence A, bounds a member of such a class.
- There cannot be a universal $\langle A\rangle$-test. Otherwise, we would have a nonempty $\Pi^0_1\langle A\rangle$ class containing only $\langle A\rangle$-randoms.
- It is also not hard to prove that (III) $\Rightarrow$ (II) (nontrivially) for A.

An interesting quasiminimal example

		(II) $\not\Rightarrow$ (I)	(II) $\Rightarrow$ (I) trivially	(II) $\Rightarrow$ (I) nontrivially
Universal $\langle A\rangle$-test	(III) $\not\Rightarrow$ (II)	✓	✓	✓
	(III) $\Rightarrow$ (II) trivially	✓	✓	?
	(III) $\Rightarrow$ (II) nontrivially	✗		
No universal $\langle A\rangle$-test	(III) $\not\Rightarrow$ (II)	✓	✗	
	(III) $\Rightarrow$ (II) trivially	✗		
	(III) $\Rightarrow$ (II) nontrivially	✓		

- For this case, we built an A with the partial computable extension property such that $\langle A\rangle$-randomness is equivalent to 3-randomness.

An interesting quasiminimal example

Proposition. For every X, there is an X-c.e. set $A \subseteq 2^{<\omega}$ such that

(1) $[A]$ is a universal Kučera X-test, and

(2) A is c.e. or has quasiminimal enumeration degree. (In fact, A has the partial computable extension property.)

- By (1), $[A]$ is a Kučera $\langle A \rangle$-test that contains every non-X-random, so every $\langle A \rangle$-random sequence is X-random.
- Also $A \leq_e X \oplus \overline{X}$, so every X-random sequence is $\langle A \rangle$-random.
- Therefore, $\langle A \rangle$-randomness is equivalent to X-randomness, and $[A]$ is a universal Kučera $\langle A \rangle$-test.

If we take $X = \emptyset''$, for example, we get an A with the PCEP such that $\langle A \rangle$-randomness is equivalent to 3-randomness. As promised, (III) $\not\Rightarrow$ (II), (II) $\Rightarrow$ (I) trivially for A, and there's a universal $\langle A \rangle$-test.

Finally, what about when (III) $\not\Rightarrow$ (II) $\not\Rightarrow$ (I)?

		(II) $\not\Rightarrow$ (I)	(II) $\Rightarrow$ (I) trivially	(II) $\Rightarrow$ (I) nontrivially
Universal $\langle A \rangle$-test	(III) $\not\Rightarrow$ (II)	✓	✓	✓
	(III) $\Rightarrow$ (II) trivially	✓	✓	?
	(III) $\Rightarrow$ (II) nontrivially	✗		
No universal $\langle A \rangle$-test	(III) $\not\Rightarrow$ (II)	✓	✗	
	(III) $\Rightarrow$ (II) trivially	✗		
	(III) $\Rightarrow$ (II) nontrivially	✓		

- If A is from ○ and B is sufficiently generic, then $A \oplus B$ is here.
- I only know a somewhat difficult construction for this case.

Thank you!