The Posner–Robinson theorem in the enumeration degrees

Mariya I. Soskova

University of Wisconsin–Madison

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The Posner-Robinson Theorem

Theorem (Posner, Robinson 1981)

Let $\mathcal{A}$ be a collection of degrees uniformly computable from $\mathbf{0}'$ and not containing $\mathbf{0}$. There is a degree $\mathbf{g}$ so that for every degree $\mathbf{a} \in \mathcal{A}$ we have that $\mathbf{a} \vee \mathbf{g} = \mathbf{g}' = \mathbf{0}'$.

1. The exact phrasing involves a cone avoidance component;
2. Slaman and Steel (1988) use a version to prove a special case of Martin's Conjecture;
3. Slaman and Shore (1999) use a version of this theorem to prove the definability of the Turing jump;
4. Kihara, Gregoriades, Ng (2021) use a version of this theorem to prove results related to the decomposability conjecture in descriptive set theory.

The enumeration degrees

Enumeration reducibility arises from the need to extend our model of relative computation to partial oracles. It was introduced independently several times including by Uspensky 1955, Rogers 1957, Myhill 1961, Selman 1971.

Definition (Friedberg and Rogers 1959, Selman 1971)

A is *enumeration reducible to* ($\leq_e$) B if and only if there is a c.e. set Γ of axioms $\langle x, D\rangle$, so that $x \in A$ if and only if for some $\langle x, D\rangle \in \Gamma$ we have $D \subseteq B$. We write $A = \Gamma(B)$.

Equivalently, $A \leq_e B$ if and only if any Turing oracle that can enumerate B can also enumerate A.

The induced structure $\mathcal{D}_e$ is the partial order of the enumeration degrees.

The Turing degrees $\mathcal{D}_T$ embed into $\mathcal{D}_e$ as the *total enumeration degrees* by $\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A \oplus \overline{A})$.

The skip operator

Let $\{\Gamma_e\}$ be an effective listing of all enumeration operators and set $K_A = \{e \mid e \in \Gamma_e(A)\}$. Note, that $K_A \equiv_e A$.

Definition

The *skip* of A is the set $A^\Diamond = \overline{K_A}$.

1. It is monotone: $A \leq_e B$ iff $K_A \leq_1 K_B$ iff $\overline{K_A} \leq_1 \overline{K_B}$;
2. It is strict: $\overline{K_A} \not\leq_e A$;
3. It agrees with the Turing jump: for all A we have that $(A \oplus \overline{A})^\Diamond \equiv_e A' \oplus \overline{A'}$.

Definition

The skip of $\mathbf{d}_e(A)$ is $\mathbf{d}_e(A)^\Diamond = \mathbf{d}_e(\overline{K_A})$.

Proposition

If $\mathbf{a} = \iota(\mathbf{x})$ is a total enumeration degree then $\mathbf{a}^\Diamond = \iota(\mathbf{x}')$.

Skip properties

For most sets A we have that $A \not\leq_e A^\Diamond$.

Theorem (AGKLMSS 2021)

$A \leq_e A^\Diamond$ if and only if A has *cototal degree*.

Cooper (1984) defined the *enumeration jump* of a set A as $K_A \oplus \overline{K_A}$.

Theorem (AGKLMSS 2021)

If $\mathbf{a} \geq \mathbf{0}'_e$ then there is some $\mathbf{b}$ so that $\mathbf{b}^\Diamond = \mathbf{a}$.

The double skip is a set-monotone operator. A simple application of the Knaster-Tarski fixed point theorem shows that:

Theorem (AGLMSS 2021)

There is a set A so that $A^{\Diamond\Diamond} = A$.

We call such sets *skip 2-cycles*. They are above every hyperarithmetical set.

Kalimullin pairs

Kalimullin 2003 isolated a definable class of pairs of degrees, later called $\mathcal{K}$-*pairs*. He used them to prove that the enumeration jump is first order definable in $\mathcal{D}_e$.

Definition (Kalimullin 2003)

A pair of sets $\{A, B\}$ is a $\mathcal{K}$-*pair* if there is a c.e. set W so that:

$$A \times B \subseteq W \text{ and } \overline{A} \times \overline{B} \subseteq \overline{W}.$$

Theorem (Kalimullin 2003)

A pair of sets $\{A, B\}$ is a $\mathcal{K}$-pair if and only if the degrees $\mathbf{d}_e(A) = \mathbf{a}$ and $\mathbf{d}_e(B) = \mathbf{b}$ satisfy

$$(\forall \mathbf{x})[(\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}.]$$

If $\mathbf{a}$ and $\mathbf{b}$ form a $\mathcal{K}$-pair then there can't be any $\mathbf{g}$ so that $\mathbf{a} \vee \mathbf{g} = \mathbf{b} \vee \mathbf{g} = \mathbf{g}^{\Diamond}$ because $(\mathbf{a} \vee \mathbf{g}) \wedge (\mathbf{b} \vee \mathbf{g}) = \mathbf{g}$.

This is the only obstacle

Theorem (Gura(2019) following Jockusch and Shore(1984))

If A and B do not form a $\mathcal{K}$-pair then there is a total function g so that $A \oplus g \equiv_e B \oplus g \equiv_e g'$.

Proof.

Fix such A and B. We build $g = \bigcup_s \sigma_s$ wehere $\sigma \in \omega^{<\omega}$:

- At stage s we start by setting $\sigma = \sigma_s\hat{\ }x\hat{\ }y$ where x, y are the s-th elements of A and B respectively.
- Consider the c.e. set $W = \{\langle a, b\rangle \mid (\exists\tau)[\tau \succeq \sigma\hat{\ }\langle a, b\rangle \ \& \ s \in W_s^\tau]\}$.
- Fix $\langle a, b\rangle$ witnessing $A \times B \not\subseteq W$ or $\overline{A} \times \overline{B} \not\subseteq \overline{W}$.
- If $\langle a, b\rangle \in W$ then let σ_{s+1} be the least $\tau \succeq \sigma\hat{\ }\langle a, b\rangle$ with $s \in W_s^\tau$.
- Otherwise set $\sigma_{s+1} = \sigma\hat{\ }\langle a, b\rangle$.

If f is any enumeration of A or B then f and g together can recover the construction and compute g'. □

$\mathcal{K}$-tuples

Definition

A finite sequence of sets $A_1 \ldots A_n$ is a $\mathcal{K}$-tuple if and only if there is a c.e. set W so that

$$A_1 \times \cdots \times A_n \subseteq W \text{ and } \overline{A} \times \cdots \times \overline{A_n} \subseteq \overline{W}$$

Theorem

Let $\mathcal{A}$ be a countable sequence of enumeration degrees that does not contain $\mathbf{0}_e$ or any $\mathcal{K}$-tuple. Then there is a total enumeration degree $\mathbf{g}$ so that for all $\mathbf{a} \in \mathcal{A}$ we have that $\mathbf{a} \vee \mathbf{g} = \mathbf{g}'$.

The Slaman–Shore theorem

Fix $n \geq 1$.

Theorem (Slaman, Shore 1999)

A is not Δ^0_n if and only if there is a set G so that $A \oplus G \equiv_T G^{(n)}$.

Proof flavor:
If A is not Δ^0_n then the proof uses Kumabe-Slaman forcing to build G so that $A \oplus G \geq_T G^{(n)}$.

An application of the iterated and relativized jump inversion theorem then finds $F \geq_T G$ so that $A \oplus F = F^{(n)}$. $\square$

We want a similar method to use for the skip operator.

A forcing style proof

Theorem (Slaman, Soskova 2025)

If A is not c.e. then there is a set G so that $A \oplus G \geq_e G^\Diamond$

Proof.

We build G so that $G(A) = \{x \mid (\exists D)[\langle x, D\rangle \in G \,\& D \subseteq A]\} = G^\Diamond$.

At stage s we have built a finite set of axioms G_s and a finite set F_s. We have committed that if $\langle z, D\rangle$ is added to G then $F_s \subseteq D$. We want to determine whether $s \in G^\Diamond$ i.e. whether $s \notin \Gamma_s(G)$.

Case 1: Suppose that there is an axiom $\langle s, E\rangle \in \Gamma_s$ so that every element in $E \setminus G_s$ has the form $\langle z, D\rangle$ with $F_s \subseteq D$ and $D \nsubseteq A$. Then let $G_{s+1} = G_s \cup E$ and $F_{s+1} = F_s$.

Case 2: Otherwise, if $\langle s, E\rangle \in \Gamma_s$ and every element $\langle z, D\rangle$ in $E \setminus G_s$ has $F_s \subseteq D$ then there is at least one $\langle z, D\rangle \in E \setminus G_s$ with $D \subseteq A$. □

A forcing style proof

Theorem (Slaman, Soskova)

If A is not c.e. then there is a set G so that $A \oplus G \geq_e G^{\Diamond}$

Proof.

Case 2: Otherwise, if $\langle s, E\rangle \in \Gamma_s$ and every element $\langle z, D\rangle$ in $E \setminus G_s$ has $F_s \subseteq D$ then there is at least one $\langle z, D\rangle \in E \setminus G_s$ with $D \subseteq A$.

Say that a is *essential* if for some $\langle s, E\rangle \in \Gamma_s$ as above we have that a appears in each D.

The set of essential elements is c.e. and a subset of A.

Pick $a \in A$ to be nonessential and add a to F_{s+1}. This guarantees that $s \in G^{\Diamond}$, so set $G_{s+1} = G_s \cup \{\langle s, F_{s+1}\rangle\}$. □

A surprise

Theorem (Slaman, Soskova)

If A is not c.e. then there is a set G so that $A \oplus G \geq_e (G^\Diamond)^\Diamond$.

Proof.

A more careful analysis of the previous construction. □

Note that G cannot be of total degree $\mathbf{d}_e(g)$ because if $A \leq_e \mathbf{0}'_e$ then $A \oplus g \leq_e g' <_e g''$!

But recall that there are G with $G = (G^\Diamond)^\Diamond$. So one (trivial) way to prove this is to build G as a skip-2-cycle above A.

Skip inversion

Recall that one of the steps in the proof of the Slaman-Shore theorem was relativized jump inversion. Does skip inversion relativize?

Theorem (AGKLMSS 2021)

For every $\mathbf{x}$ and every degree $\mathbf{a} \geq \mathbf{x}'$ there is a degree $\mathbf{y} \geq \mathbf{x}$ with $\mathbf{y}^{\Diamond} = \mathbf{a}$.

Theorem (Slaman, Soskova 2025)

However, there are degrees $\mathbf{x}$ and $\mathbf{a} \geq \mathbf{x}^{\Diamond}$ so that no degree $\mathbf{y} \geq \mathbf{x}$ has $\mathbf{y}^{\Diamond} = \mathbf{a}$.

Failure of relativized skip inversion

Theorem (Slaman, Soskova 2025)

There are degrees $\mathbf{x}$ and $\mathbf{a} \geq \mathbf{x}^{\Diamond}$ so that no degree $\mathbf{y} \geq \mathbf{x}$ has $\mathbf{y}^{\Diamond} = \mathbf{a}$.

Proof.

$A \leq_e Y^{\Diamond}$ implies that there is some $B \leq_e Y$ so that if $x \in A$ then $B^{[x]}$ is finite and if $x \notin A$ then $B^{[x]} = \omega$.

$$C = \begin{cases} \omega, & \exists n[|X \cap B^{[2n]}| > n]\,; \\ \emptyset, & \text{otherwise.} \end{cases}$$

$C \leq_e X \oplus B \leq_e Y$ so $\overline{C} \leq_e Y^{\Diamond}$. Let $\overline{C} = \Gamma(A)$.

Case 1: We add some finite $D \in A$ so that $\Gamma(A) \neq \emptyset$ implying $C = \emptyset$. We leave some $2n > \max D$ out of A so that $B^{[2n]} = \omega$ and so $C = \omega$.

Case 2: $\Gamma(A)$ must be empty. We put all even numbers in A forcing every even column of B to be finite, say $B^{[2n]}$ bounded by b_n. We then commit to making X sparse enough to get a contradiction: $x_0 > b_0$, $x_1 > b_1$, … □

Final result

Theorem (Slaman, Soskova 2025)

If A is not c.e. then there is a set G so that $A \oplus G \equiv_e G^\Diamond \oplus (G^\Diamond)^\Diamond$.

Proof.

We build $G = G_1 \oplus G_2 \oplus G_3$ so that

1. $G_1(A) = G^\Diamond$;
2. $G_2(A) = (G^\Diamond)^\Diamond$;
3. $G_3(\emptyset^\Diamond) = A$.

We adapt the notion of nonessential element using the fact that $(A, A, \emptyset^\Diamond)$ cannot be a $\mathcal{K}$-tuple. □

Question

How do we iterate this construction for higher n?

Thank you

Gian-Carlo Rota

According to ...[the "one-shot"] ... view, mathematics would consist of a succession of targets, called problems, which mathematicians would be engaged in shooting down by well-aimed shots. But where do problems come from, and what are they for? If the problems of mathematics were not instrumental in revealing a broader truth, then they would be indistinguishable from chess problems or crossword puzzles. Mathematical problems are worked on because they are pieces of a larger puzzle.

Ivan Soskov

The good puzzles are the ones that will never be completely solved.