Effective second countability in computable analysis

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Contents

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Computable topology

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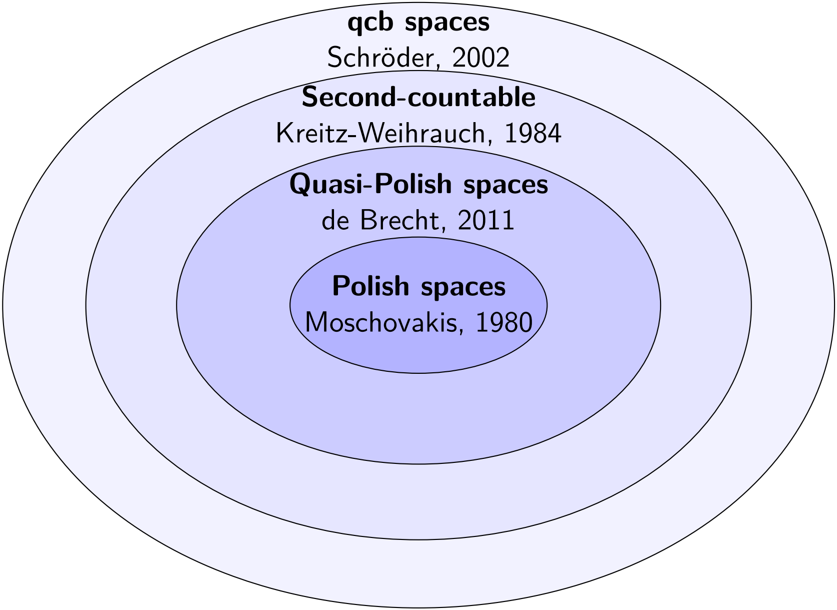
- One of the modern approaches to computable topology is based on Kreitz and Weihrauch's **theory of representations**.
- It was used to study computability on a wide range of families of topological spaces.

Some families of topological spaces



Computable definitions associated

Each classical notion should come with its associated "effective version".

Computable Polish spaces

Computable Polish spaces were introduced by Moschovakis in 1980.

Quasi-Polish spaces

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- de Brecht, Pauly and Schröder: Overt choice, *Computability* 2020.

Building on (and answering problems from):

- Selivanov: Towards the Effective Descriptive Set Theory, CiE 2015.
- Korovina and Kudinov: On Higher Effective Descriptive Set Theory, CiE 2017.

Schröder's qcb_0 spaces

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We will come back to it.

Second countable spaces

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- Selivanov: Towards the Effective Descriptive Set Theory, CiE 2015.
- Neumann, Pauly, Pradic, Valenti, Computably discrete represented spaces, CiE 2025.

Example taken from Computable metrization, Grubba, Schröder, Weihrauch, MLQ 2007

Definition 4.2 (Computably regular space) A *computably regular space* is a computable T_0-space $\boldsymbol{X}$ such that there is a computable function $t_3 :\subseteq \Sigma^* \times \Sigma^* \longrightarrow \Sigma^\omega$ such that $R := \mathrm{dom}(t_3)$ is r. e.,

$$(\forall v \in \mathrm{dom}(\nu))(\nu(v) = \bigcup_{(u,v)\in R} \nu(u)), \quad \text{and} \quad (\forall (u,v) \in R)(\nu(u) \subseteq \psi^{\mathrm{un}}(t_3(u,v)) \subseteq \nu(v)).$$

Every computably regular space is regular.

Definition 4.3 (Computably normal) A *computably normal space* is a computable T_0-space $\boldsymbol{X}$ such that the multi-function $t_4 :\subseteq \tau^{\mathrm{c}} \times \tau^{\mathrm{c}} \rightrightarrows \tau \times \tau$ defined by

$$\mathrm{dom}(t_4) := \{(A,B) \in \tau^{\mathrm{c}} \times \tau^{\mathrm{c}} \mid A \cap B = \emptyset\},$$
$$(O_A, O_B) \in t_4(A,B) :\Leftrightarrow A \subseteq O_A \wedge B \subseteq O_B \wedge O_A \cap O_B = \emptyset,$$

is $(\psi^{\mathrm{un}}, \psi^{\mathrm{un}}, \theta^{\mathrm{un}}, \theta^{\mathrm{un}})$-computable.

Every second-countable regular space is normal [4]. We prove the computable version.

Theorem 4.4 *Every computably regular space is computably normal.*

Effective second countability

- There seemed to be a lack of a systematical study of the effective versions of the statement

  “X has a countable basis $(B_i)_{i\in\mathbb{N}}$”

  that we have tried to address.

Contents

Represented spaces, realizer

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A **realizer** for a multi-function $f : (X, \rho) \rightrightarrows (Y, \tau)$ between represented spaces is a partial map $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

$$\forall p \in \mathrm{dom}(\rho),\ \tau(F(p)) \in f(\rho(p)).$$

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A multi-function is **computable** if it has a computable realizer, and **continuously realizable** if it has a continuous realizer.

Continuous and computable translation

Definition

If ρ and τ are representations of X, we put $\rho \leq \tau$, and say that ρ **translates to** τ, if the identity $\mathrm{id} : (X, \rho) \to (X, \tau)$ is computable.

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The relations $\leq$ and $\leq_t$ induce equivalence relations.

Admissibility

Definition (Weihrauch-Kreitz, 1985)

A representation δ of a topological space X is **admissible** if it is continuous $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ and if for any continuous representation $\tau :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ we have $\tau \leq_t \delta$.

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Theorem (Weihrauch-Kreitz, 1985)

A representation δ of a topological space Y is admissible if and only if for every represented space (X, ρ) equipped with the final topology of its representation and every function $f : X \to Y$, we have the equivalence:

- *f is continuous,*
- *f is continuously realizable.*

Admissible representation theorem for second countable spaces

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The **standard representation** of X associated to $(B_i)_{i\in\mathbb{N}}$ is given by

$$\rho((u_n)_{n\in\mathbb{N}}) = x \iff \{u_n \mid n \in \mathbb{N}\} = \{k \in \mathbb{N} \mid x \in B_k\}.$$

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3. *All admissible representations of a second countable space are continuously equivalent to a standard representation.*

What about computability?

Schröder's PhD, 2002

Presents a general approach to computable topology based on the representation of the Sierpiński space.

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The Sierpiński space $\mathbb{S}$ is $\{0, 1\}$ with $\{1\}$ open and $\{0\}$ not open.

Representation of $\mathbb{S}$

The usual representation $c_{\mathbb{S}} : \mathbb{N}^{\mathbb{N}} \to \mathbb{S}$ of $\mathbb{S}$ is given by

$$c_{\mathbb{S}}(0^\omega) = 0,$$

$$c_{\mathbb{S}}(u) = 1 \text{ for } u \neq 0^\omega.$$

We get a representation

The representation $c_{\mathbb{S}}$ is admissible, and thus for any represented space X equipped with the final topology of its representation, every continuous map $f : X \to \mathbb{S}$ has a continuous realizer.

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Example

$\mathcal{O}(\mathbb{N}) = \mathcal{P}(\mathbb{N})$, the set of subsets of $\mathbb{N}$ equipped with the Scott topology. Subsets of $\mathbb{N}$ are described by enumerations.

Admissibility theorem

Theorem (Schröder, 2002)

A representation is admissible if and only if

$$\begin{aligned} X &\hookrightarrow \mathcal{O}\mathcal{O}(X) \\ x &\mapsto \{O \mid x \in O\} \end{aligned}$$

has a continuously realizable inverse.

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has a computable inverse.

Computably admissible representation

Theorem (Schröder, 2002)

A representation ρ of a set Y is computably admissible iff for every represented space X and every function $f : X \to Y$, we have

$$f \text{ is computable} \iff f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X) \text{ is computable.}$$

Computably admissible representation as Computably Kolmogorov spaces

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- A topological space is T_0 when points are uniquely determined by the open sets to which they belong.
- A represented space is CT_0 when the name of a point can be computed from a name of the set of open sets to which it belongs.

Coming back to the Weihrauch-Kreitz Theorem

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Theorem (Weihrauch-Kreitz, 1985)

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A computable Weihrauch-Kreitz Theorem?

Fact

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2. *The standard representations of a second countable space* ***are not*** *all computably equivalent.*

A computable Weihrauch-Kreitz Theorem?

Fact

1. *All standard representations are computably admissible.*
2. *The standard representations of a second countable space* ***are not*** *all computably equivalent.*
3. *A computably admissible representation of a second countable space* ***does not*** *have to be computably equivalent to a standard representation.*

Computable second countability

Definition

We say that (X, ρ) is **computably second countable** when ρ is computably equivalent to a standard representation.

Goal of today's talk

- The fact that the notion of *computably second countable space* is very robust does not really need justification.

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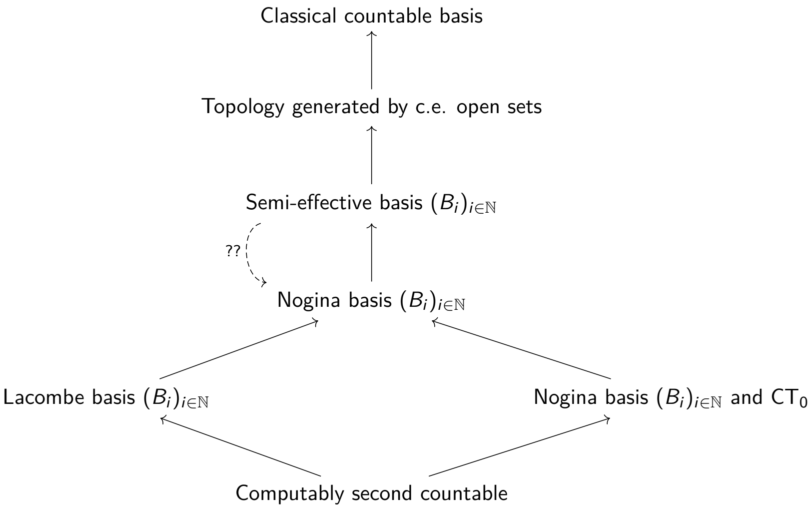
- The fact that the notion of *computably second countable space* is very robust does not really need justification.
- But we still gather useful equivalent definitions.
- Describe a whole range of weak forms of effective second countability.
- Emphasize the fact that Schröder's work, whose main goal is often understood as extending the work of Weihrauch to non second-countable spaces, is also useful for second-countable spaces.

Different forms of effective second countability



Computably second countable spaces

X equipped with a standard representation

$\updownarrow$

Lacombe basis $(B_i)_{i \in \mathbb{N}}$ and CT_0

$\updownarrow$

X computably embeds in $\mathcal{O}(\mathbb{N})$

$\updownarrow$

Computably open representation and CT_0

Contents

Bases of c.e. open sets.

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The **c.e. open sets** of a represented space (X, ρ) are the computable points of $\mathcal{O}(X)$.
They are the semi-decidable sets.

The weakest form of effective second countability asks that the c.e. open sets form, classically, a basis of the topology of X.

Semi-effective basis

Definition

A **semi-effective basis** for (X, ρ) is a computable map

$$B : \mathbb{N} \to \mathcal{O}(X)$$

whose image forms a basis.

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Thus the elements are constructively open, they can be enumerated, but they form a basis only classically.

Nogina Basis

Definition

A **Nogina basis** for (X, ρ) is a semi-effective basis $(B_i)_{i\in\mathbb{N}}$ for which the map

$$\begin{aligned} X \times \mathcal{O}(X) &\rightrightarrows \mathbb{N} \\ (x, U) &\mapsto \{i,\, x \in B_i \subseteq U\} \end{aligned}$$

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References

Elena Yu. Nogina. *Effectively topological spaces*. Doklady Akademii Nauk SSSR, 1966.

Gregoriades, Kispéter and Pauly. *A comparison of concepts from computable analysis and effective descriptive set theory*. Mathematical Structures in Computer Science, 2016.

Lacombe basis

If $(B_i)_{i\in\mathbb{N}}$ is a semi-effective basis of X, then the following map is computable and onto:

$$\begin{aligned}\mathcal{O}(\mathbb{N}) &\to \mathcal{O}(X)\\ A &\mapsto \bigcup_{i\in A} B_i.\end{aligned}$$

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In words: the open set can uniformly be written as countable unions of basic sets.

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Daniel Lacombe. Quelques procédés de définition en topologie récursive. In Arend Heyting, editor, Constructivity in mathematics, Proceedings of the colloquium held at Amsterdam, 1957.

Klaus Weihrauch and Tanja Grubba. Elementary computable topology. J. Univers. Comput. Sci., 2009

Computably open representation

Definition

A representation $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is **computably open** if the map

$$\hat{\rho} : \mathcal{O}(\mathbb{N}^{\mathbb{N}}) \to \mathcal{O}(X)$$
$$U \mapsto \rho(U)$$

is well defined and computable.

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Another name for this notion is **computably fiber-overt representation**.

Embedding into $\mathcal{O}(\mathbb{N})$

Definition

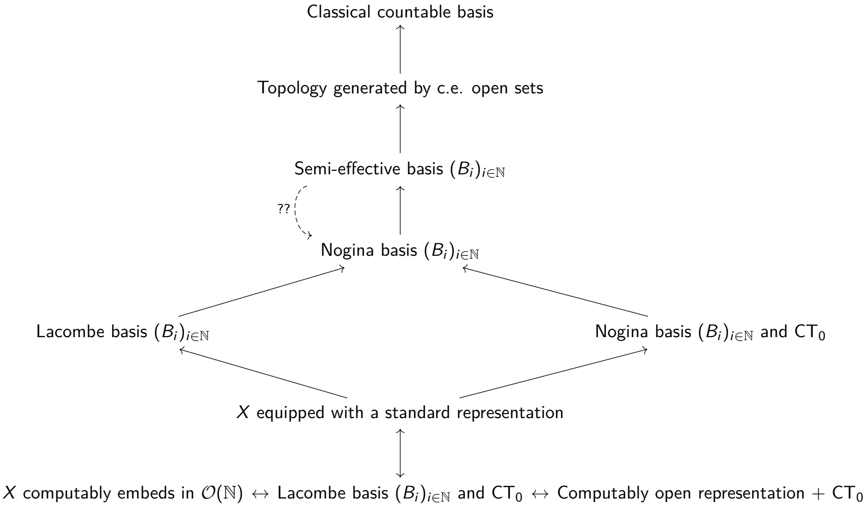
A **computable embedding** between represented spaces (X, ρ) and (Y, δ) is a computable injection $f : X \hookrightarrow Y$ with a computable inverse $g : \mathrm{Im}(f) \hookrightarrow X$.

Main theorem

Theorem (Brattka, R.)

All implications between the notions are shown on the following figure. (There is one conjecture.)

All the implications



Most relevant implication

Theorem

If X is equipped with the standard representation associated to $(B_i)_{i\in\mathbb{N}}$, then $(B_i)_{i\in\mathbb{N}}$ is a Lacombe basis.

Type 2 Moschovakis Theorem

This implication is a Type 2 version of a theorem from :
Moschovakis, Recursive metric spaces, *Fundamenta Mathematicae* 1964

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Generalizations due to Dieter Spreen (1998).

Intrinsic embedding

Corollary

If X is computably second countable and $Y \subseteq X$, then Y equipped with the restriction of the representation of X is computably second countable as well, and the map

$$\begin{aligned} \mathcal{O}(X) &\to \mathcal{O}(Y) \\ U &\mapsto U \cap Y \end{aligned}$$

is onto and has a computable multivalued inverse.

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In the vocabulary of Bauer, any $Y \subseteq X$ is an **intrinsic subset** of X *(Spreen spaces and the synthetic Kreisel-Lacombe-Shoenfield-Tseitin theorem. JLA, 2025.)*

Computably sequential embedding

We call it a **computably sequential embedding**.

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Indeed, for $Y \subseteq X$, the map

$$\begin{aligned} \mathcal{O}(X) &\to \mathcal{O}(Y) \\ U &\mapsto U \cap Y \end{aligned}$$

is onto exactly when the subset topology $\{U \cap Y \mid U \in \mathcal{O}(X)\}$ is sequential (Schröder).

New version

The above corollary is seen as an effective version of "a second-countable space is hereditarily sequential".

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Corollary

All subsets of a computably second countable space are computably sequential.

A non intrinsic embedding

Example (Friedberg 1958, Bauer 2025)

Consider the map

$$\begin{aligned} \mathbb{N}^{\mathbb{N}} &\to \mathcal{O}(\mathbb{N}) \\ (u_n)_{n\in\mathbb{N}} &\mapsto \{\langle n, u_n\rangle \mid n \in \mathbb{N}\}. \end{aligned}$$

In Type 1 computability (Markovian constructivism), it is **not** an intrinsic embedding.

Weihrauch-Grubba approach

Unifying result

We also clarify the relationship between different approaches to computable topology.

Weihrauch-Grubba approach

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We also clarify the relationship between different approaches to computable topology.
In particular, we can understand the relationship with the Weihrauch-Grubba approach based on a notion of computable presentation.

Weihrauch-Grubba and Schröder approaches

Definition

A *computable topological space* is a pair $(X, (B_i)_{i\in\mathbb{N}})$, where X is a set and $(B_i)_{i\in\mathbb{N}}$ is the basis of a T_0 topology on X for which there exists a computable function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for any i, j in $\mathbb{N}$:

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Consider also the standard representation associated to $(B_i)_{i\in\mathbb{N}}$.

Weihrauch-Grubba and Schröder approaches

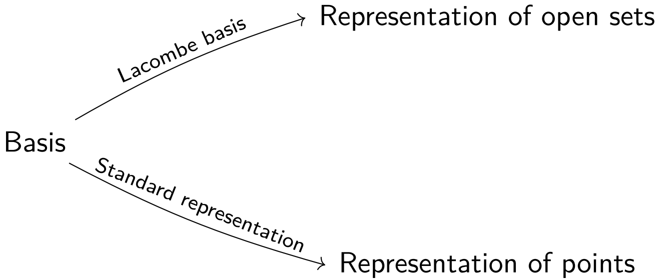

Figure: Weihrauch-Grubba approach

Weihrauch-Grubba and Schröder approaches

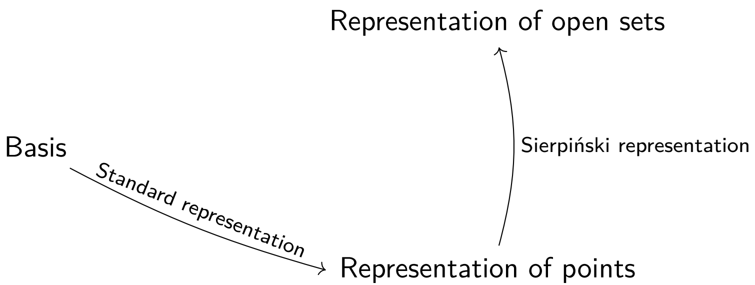

Figure: Sierpiński representation approach

Weihrauch-Grubba and Schröder approaches

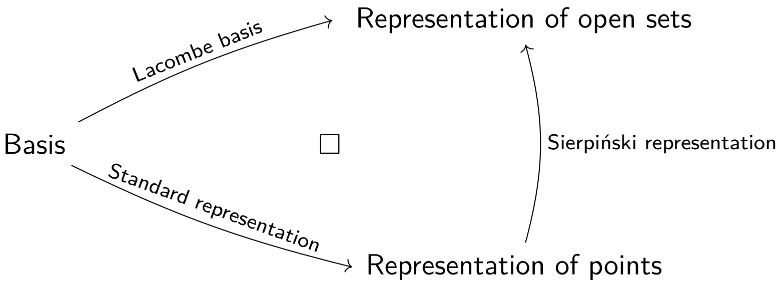

Figure: Compatibility of the approaches

Contents

Non computably second countable copy of $\mathbb{S}$

Replace the standard representation of the Sierpiński space

$$\begin{aligned} c_{\mathbb{S}} : \mathbb{N}^{\mathbb{N}} &\to \mathbb{S} \\ u &\mapsto 0 \text{ if } u = 0^{\omega} \\ &\mapsto 1 \text{ if } u \neq 0^{\omega} \end{aligned}$$

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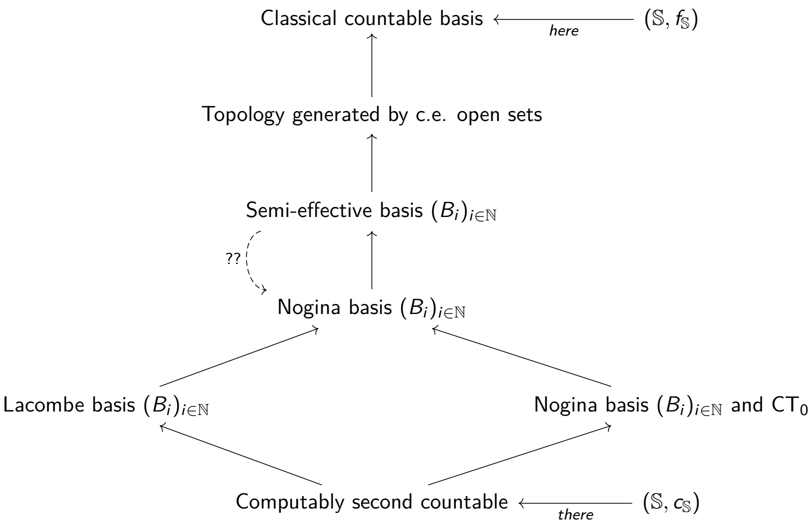
Hoyrup and Rojas, On the information carried by programs about the objects they compute, *Theory of Computing Systems*, 2016

Where are we?


Classical countable basis
here
$(\mathbb{S}, f_{\mathbb{S}})$
Topology generated by c.e. open sets
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??
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Computably second countable
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A second example

Start with a classical example of a sequential but not hereditarily sequential topological space.

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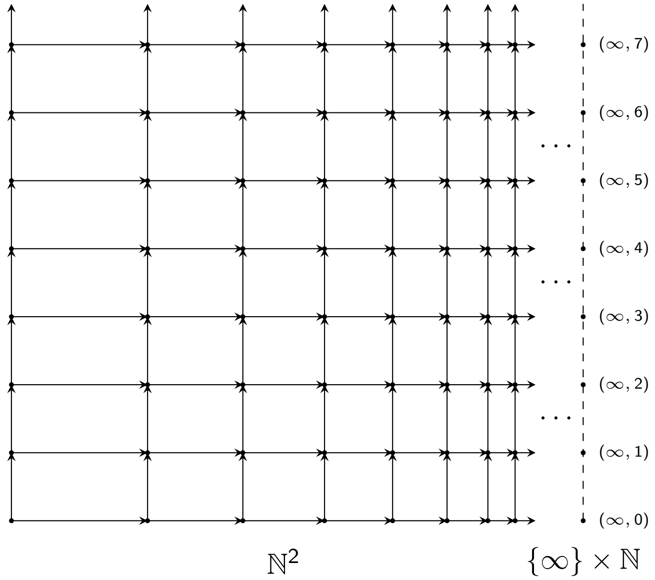
Start with a classical example of a sequential but not hereditarily sequential topological space.

Take

$$X = \mathbb{N}^2 \cup \{\infty\} \times \mathbb{N},$$

with topology discrete on $\mathbb{N}^2$, discrete on $\{\infty\} \times \mathbb{N}$, and $(n, p) \underset{n\to\infty}{\to} (\infty, p)$.

Illustration of the example



Add (∞, ∞)

- Add a new point (∞, ∞), with neighborhood basis defined as the set of sets of the form

$$B_{f,k} = \{(n, m) \in (\mathbb{N} \cup \{\infty\}) \times \mathbb{N} \mid n > f(m), m > k\} \cup \{(\infty, \infty)\},$$

for $f \in \mathbb{N}^{\mathbb{N}}$, $k \in \mathbb{N}$.

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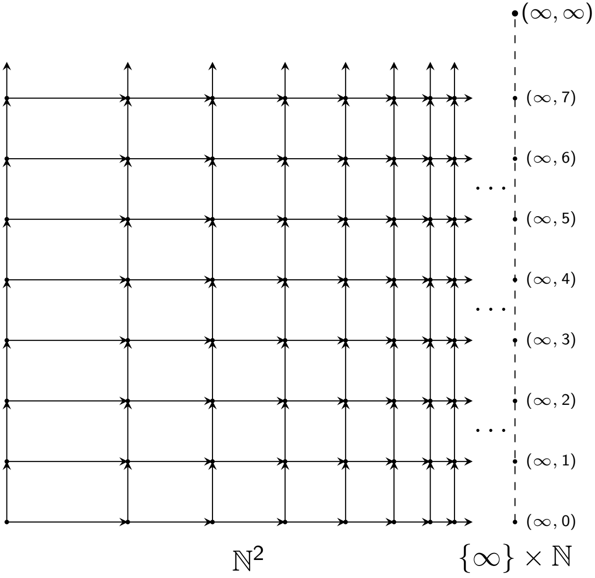
The closure of $\mathbb{N}^2$ in X is all of X.

The sequential closure of $\mathbb{N}^2$ is $X \setminus (\infty, \infty)$: for a sequence in $\mathbb{N}^2$ to converge to (∞, ∞), the first component should grow faster than all functions.

Drawing


(∞,∞)
$(\infty,7)$
$(\infty,6)$
$(\infty,5)$
$(\infty,4)$
$(\infty,3)$
$(\infty,2)$
$(\infty,1)$
$(\infty,0)$
$\mathbb{N}^2$
$\{\infty\}\times\mathbb{N}$

Example

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We consider an effective version of the above: we replace the basis

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The topological space Y thus obtained is now second countable, and thus hereditarily sequential.
However, a *computable* sequence of elements of $\mathbb{N}^2$ cannot converge to (∞, ∞).

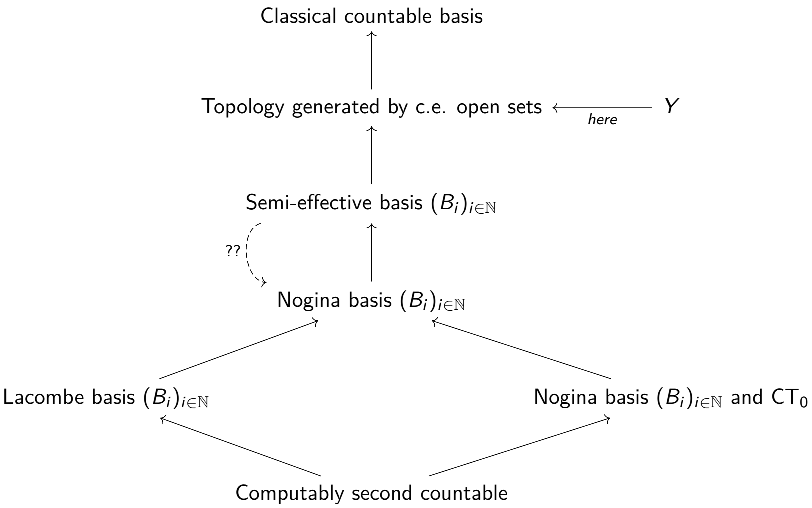
Conclusion

We thus obtain a represented space in which the c.e. open sets do generate the topology, but in which no computable sequence of c.e. open sets can be a basis: the image of a computable map

$$\mathbb{N} \to \mathcal{O}(Y)$$

is never a basis.

Where are we?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Computably second countable

Contents

Setting

If a topological space X is second countable and A, then B. (1)

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If a topological space X is second countable and A, then B. (1)

Which form of effective second countability is necessary to get an effective version of this statement?

Urysohn Metrization

Theorem (Urysohn 1925, Tychonoff 1926)

The following are equivalent for a second countable space X:

1. *X (topologically) embeds into the Hilbert cube $[0,1]^{\mathbb{N}}$,*
2. *X is regular and T_0,*
3. *X is metrizable.*

Regularity

A topological space X is **regular** if for every point x and closed A with $x \notin A$, there are U_1 and U_2 disjoint open with $x \in U_1$ and $F \subseteq U_2$.

Computable regularity

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A topological space X is **regular** if for every point x and closed A with $x \notin A$, there are U_1 and U_2 disjoint open with $x \in U_1$ and $F \subseteq U_2$.

Computable regularity

A represented space X is **computably regular** if the following multi-function is well defined and computable:

$$R :\subseteq X \times \mathcal{A}_-(X) \rightrightarrows \mathcal{O}(X)^2$$
$$(x, A) \mapsto \{(U, V), x \in U \,\&\, A \subseteq V \,\&\, U \cap V = \emptyset\}.$$

Strong computable regularity

Strong computable regularity (Schröder, 1998)

A represented space X is **strongly computably regular** if the following multi-function is well defined and computable:

$$P : \mathcal{O}(X) \rightrightarrows \mathcal{O}(X)^{\mathbb{N}} \times \mathcal{A}_{-}(X)^{\mathbb{N}}$$

$$O \mapsto \{(U_n, V_n)_{n\in\mathbb{N}}, \forall n \in \mathbb{N}, U_n \subseteq V_n \subseteq O, O = \bigcup_{n\in\mathbb{N}} U_n\}.$$

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This includes a version of the Lindelöf property, it could maybe be called "computably regular-Lindelöf".

Schröder-Urysohn Effective Metrization

Theorem

The following are equivalent for a represented space (X, ρ)*:*

1. (X, ρ) *computably embeds into the Hilbert cube* $[0,1]^{\mathbb{N}}$*,*
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Computable second countability is necessary.

Second example

Fact

A second countable space is separable.

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A second countable space is separable.

Proof.

Consider a countable basis (B_i). The set $\{B_i \mid B_i \neq \emptyset\}$ is also countable. Then apply choice. $\square$

Computable separability

A represented space X is **computably separable** if there exists a dense and computable sequence, i.e. a computable map

$$f : \mathbb{N} \to X$$

with dense image.

Naive version

Open choice:

$$\begin{aligned} OC : \mathcal{O}(X) \setminus \{\emptyset\} &\rightrightarrows X \\ O &\mapsto O. \end{aligned}$$

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Effective Fact (?)

Let (X, ρ) be a represented space which admits a semi-effective basis of non-empty sets and that has a computable open choice problem. Then (X, ρ) is effectively separable.

Naive version

Open choice:

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Effective Fact (?)

Let (X, ρ) be a represented space which admits a semi-effective basis of non-empty sets and that has a computable open choice problem. Then (X, ρ) is effectively separable.

True, but useless.

Open choice

Theorem (Brattka, R.)

A represented space (X, ρ) *has computable Open Choice if and only if it is computably separable.*

Saving our fact

Non-total Open Choice:

$$OC^* : \mathcal{O}(X) \setminus \{\emptyset, X\} \rightrightarrows X$$
$$O \mapsto O.$$

The fact is saved!

Effective Fact (!)

Let (X, ρ) be a represented space which admits a semi-effective basis of non-empty and non-total sets, and that has computable Non-total Open Choice problem.
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Let (X, ρ) be a represented space which admits a semi-effective basis of non-empty and non-total sets, and that has computable Non-total Open Choice problem.
Then (X, ρ) is effectively separable.

Having a computable Non-total Open Choice problem is **not** equivalent to being computably separable.

Thank you for your attention

Is the Bordelais computably separable?