Randomness and differentiation

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Outline

Effective reals

Definition

A **computable approximation** is a computable Cauchy sequence of rationals reals.
A **Cauchy name** is a computable approximation $a_0, a_1, \dots$ that fulfills $|a_m - a_n| \leq 2^{-m}$ for every $n \geq m \geq 0$.

- **left-c.e. reals**: limit points of strictly increasing computable approximations
- **computable reals**: limit points of Cauchy names

Computable functions

- On rationals: computable by a Turing machine

Definition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is **computable** if there is a Turing functional that, for every Cauchy name $p_0, p_1, \cdots \to x \in A$ as an oracle, return a Cauchy name $q_0, q_1, \cdots \to f(x)$.

On a compact domain A, the following equivalent characterization of computability holds by Pour-El and Richards [8].

Proposition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is computable iff

- *there is a computable function $h : \mathbb{N} \to \mathbb{N}$ such that, for every n and $|x - y| \leq 2^{-h(n)}$, it holds that $|f(x) - f(y)| \leq 2^{-n}$ and*
- *for some computable sequence of reals $r_0, r_1, \ldots,$ which is dense in $[0, 1]$, the sequence $f(v_i)_{i \in \mathbb{N}}$ is computable.*

Algorithmic randomness: topological characterizations

- A **Martin-Löf test** is a uniformly effective sequence of open sets $(U_0, U_1, \ldots)$, such that the set U_n has a uniform measure of at most 2^{-n-1}. In case the (uniform) measure of U_n is exactly equal to 2^{-n-1} for every n, the Martin-Löf test $(U_0, U_1, \ldots)$ is called **Schnorr test**.
  A real α **fails** the Martin-Löf test $(U_0, U_1, \ldots)$ if $\alpha \in U_i$ **for every** i.
- A **Solovay test** is a uniformly effective sequence of intervals $(S_0, S_1, \ldots)$, such that the sum of $|S_n|$ is finite. In case the latter sum is computable, the Solovay test $(S_0, S_1, \ldots)$ is called **total**.
  A real α **fails** the Solovay test $(S_0, S_1, \ldots)$ if $\alpha \in S_i$. **for infinitely many** i.
- A real α is **Martin-Löf random** (or, shortly, **ML random**)
  $:\iff$ no Martin-Löf test fails on α
  $\iff$ no Solovay test fails on α.
- A real α is **Schnorr random**
  $:\iff$ no Schnorr test fails on α
  $\iff$ no total Solovay test fails on α.

Unpredictability and martingales

- A **martingale** is a function $M : 2^{<\omega} \to \mathbb{R}$ that fulfills for all σ the equality

$$M(\sigma 0) + M(\sigma 1) = 2M(\sigma)$$

- A **supermartingale** is a function $M : 2^{<\omega} \to \mathbb{R}$ that fulfills for all σ the inequality

$$M(\sigma 0) + M(\sigma 1) \leq 2M(\sigma)$$

- A (super)martingale M **succeeds** on a real α if $\limsup_{n\to\infty} M(A \upharpoonright n) = \infty$.

Atomless martingales $\leftrightarrow$ nondecreasing continuous functions on $[0,1]$ with $f(0) = 0$

Algorithmic randomness as unpredictability

- A real α is Martin-Löf random
  $\iff$ no c.e. (super)martingale succeeds on α.
  $\iff$ no c.e. (super)martingale satisfies $\lim_{n\to\infty} M(\alpha \restriction n) = \infty$.
- A real α is **computably random**
  $:\iff$ no computable martingale succeeds on α.
  $\iff$ no computable martingale M satisfies $\lim_{n\to\infty} M(\alpha \restriction n) = \infty$.
- a real α is Schnorr random
  $:\iff$ no computable martingale M satisfies the property

$$\exists^{\infty} n : M(\alpha \restriction f(n)) \geq n$$

  for any computable index function f.

Computable randomness and differentiability

Theorem (Brattka, Miller, Nies)

For every real $x \in (0,1)$, the following equivalence holds:

- *x is computably random $\iff$ every computable* ***nondecreasing*** *function $f : [0,1] \to \mathbb{R}$ is differentiable in x.*

Theorem (Freer, Kjos-Hanssen, Nies, Stephan)

For every real $x \in (0,1)$, the following equivalence holds:

- *x is computably random $\iff$ every computable* ***Lipschitz continuous*** *function $f : [0,1] \to \mathbb{R}$ is differentiable in x.*

Outline of the proof of the $\implies$ -part

Functions of bounded variation

Definition

Let f be a function (on rationals or reals). Then

$$V_f = \sup \left\{ \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| : \begin{cases} t_0, \ldots, t_n \in \mathrm{dom}(f) \text{ and} \\ t_0 < \cdots < t_n \end{cases} \right\} \in \mathbb{R} \cup \{\infty\}$$

is called **variation** of f.
F is a function **of bounded variation** if $V_f < \infty$.

Proposition (Jordan's theorem)

1. *A function $f : [0,1] \to \mathbb{R}$ is a function on bounded variation iff $f = g_1 - g_2$ for two nondecreasing functions $g_1, g_2 : [0,1] \to \mathbb{R}$*
2. *A function $f : [0,1]|_{\mathbb{Q}} \to \mathbb{Q}$ is a function on bounded variation iff $f = g_1 - g_2$ for two nondecreasing functions $g_1, g_2 : [0,1]|_{\mathbb{Q}} \to \mathbb{Q}$.*

Martin-Löf randomness and differentiability

Theorem (Brattka, Miller, Nies)

For every real $x \in (0,1)$, the following equivalence holds:

- *x is ML random $\iff$ every function **of bounded variation** $f : [0,1] \to \mathbb{R}$ is differentiable in x.*

Solovay reducibility on LEFT-CE

- On left-c.e. reals: measure of convergence speed of approximations

Definition

A left-c.e. real α is Solovay reducible to left-c.e. real β, written $\alpha \leq_{\mathrm{S}} \beta$ if there exists two monotone increasing computable approximations $a_0, a_1, \dots \nearrow \alpha$ and $b_0, b_1, \dots \beta$ and a constant $c > 0$ such that

$$\alpha - a_n < c(\beta - b_n)$$

Translation functions and ML randomness: the LEFT-CE case

Theorem (Barmpalias, Lewis-Pye, 2017)

Let α be a left-c.e. real and β be a ML random left-c.e. real. Then there exists a constant d such that, for every two monotone increasing approximations $a_0, a_1, \cdots \nearrow \alpha$ and $b_0, b_1, \cdots \nearrow \beta$, it holds that

$$\lim_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n} = d.$$

In particular, it implies that $\alpha \leq_{\mathrm{S}} \beta$.
Moreover, $d = 0$ iff α is ML nonrandom.

Ourline of the proof

Solovay reducibility via translation functions: general frame

Definition (T.)

Let α and β be two reals.

A computable function $f :\subseteq \mathbb{Q} \to \mathbb{Q}$ that fulfills $\mathbb{Q}|_{[0,\beta)} \subseteq \mathrm{dom}(f)$ and $\lim_{q \nearrow \beta} f(q) = \alpha$ is called $\mathbb{Q}$-translation funciton from β to α.

A computable function $f :\subseteq \mathbb{R} \to \mathbb{R}$ that fulfills $[0, \beta) \subseteq \mathrm{dom}(f)$ and $\lim_{x \nearrow \beta} f(x) = \alpha$ is called $\mathbb{R}$-translation funciton from β to α.

α is reducible to β via a ($\mathbb{Q}$- or $\mathbb{R}$-)translation function f if there exists a constant c such that

$$\limsup_{\substack{x \nearrow \beta \\ x \in \mathrm{dom}(f)}} \frac{|\alpha - f(x)|}{|\beta - x|} \leq c$$

Functional versions of Solovay reducibility

Restricting the set of admissible translation functions, we obtain equivalent characterizations of already known generalizations of Solovay reducibility outside of LEFT-CE; all of them except $\leq_{\mathrm{cL}}^{\mathrm{loc}}$ coincide with $\leq_{\mathrm{S}}$ on LEFT-CE.

- (Solovay, 1975) $\alpha \leq_{\mathrm{S}} \beta$ iff $\alpha \leq_{\mathrm{S}}^{*} \beta$ via $\mathbb{Q}$-translation function f with $f(\mathbb{Q}|_{[0,\alpha)}) \subseteq [0,\beta)$;
- (Titov, 2023) $\alpha \leq_{\mathrm{S}}^{\mathrm{m}} \beta$ iff $\alpha \leq_{\mathrm{S}} \beta$ via monotone nondecreasing f;
- (Kumabe, Miyabe, Suzuki, 2024) $\alpha \leq_{\mathrm{cL}}^{\mathrm{open}} \beta$ iff $\alpha \leq_{\mathrm{S}}^{*} \beta$ via $\mathbb{R}$-translation function f. Moreover, f can be chosen Lipschitz continuous.
- (Kumabe, Miyabe, Mizusawa, Suzuki, 2020) $\alpha \leq_{\mathrm{S}}^{\mathbb{R}} \beta$ iff $\alpha \leq_{\mathrm{cL}}^{\mathrm{open}} \beta$ via $\mathbb{R}$-translation function f with $(f[0,\alpha)) \subseteq [0,\beta)$. Moreover, f can be chosen Lipschitz continuous and nondecreasing.
- (Kumabe, Miyabe, Suzuki, 2024) $\alpha \leq_{\mathrm{cL}}^{\mathrm{loc}} \beta$ iff $\alpha \leq_{\mathrm{S}}^{\mathbb{R}} \beta$ iff $\alpha \leq_{\mathrm{cL}}^{\mathrm{open}} \beta$ via $\mathbb{R}$-translation function f with $[0,1] \subseteq \mathrm{dom}(f)$. Moreover, f can be chosen Lipschitz continuous.

Translation functions and ML-randomness: generalizations

Theorem (T., 2024)

Let α be a real and β be a ML random real. Then there exists a constant d such that, for ***monotone nondecreasing*** *$\mathbb{Q}$-translation function from β to α (if exists), it holds that*

$$\lim_{q \nearrow \beta} \frac{\alpha - f(q)}{\beta - q} = d.$$

In particular, it implies that $\alpha \leq^{\mathrm{m}}_{\mathrm{S}} \beta$.

- left differentiability in β.
- The monotonicity requirement cannot be omitted!

Translation functions and ML-randomness: generalizations (NEW)

Similar results can be achieved for other types of admissible translation functions.

Theorem

Let α be a real and β be a ML random real. Then there exists a constant d such that

- *For every $\mathbb{R}$-translation function* ***of bounded variation*** *f from β to α (if exists), it holds that*

$$\lim_{n\to\infty} \frac{|\alpha - f(q)|}{|\beta - q|} = d. \tag{1}$$

  In particular, if such function exists, then $\alpha \leq_{\mathrm{cL}}^{\mathrm{open}} \beta$. Moreover, if α is Martin-Löf nonrandom, then $d = 0$.

- *For every* ***monotone nondecreasing*** *$\mathbb{R}$-translation function from β to α (if exists), (1) holds. In particular, if such function exists, then $\alpha \leq_{\mathrm{S}}^{\mathrm{m}} \beta$. Moreover, if α is Martin-Löf nonrandom, then $d = 0$.*
- *For every $\mathbb{Q}$-translation function* ***of bounded variation*** *f from β to α (if exists), (1) holds.*

Applications: differentiability in a fixed-point

Corollary

For every real Martin-Löf random real β, the following statements hold

- *Every nondecreasing $\mathbb{R}$-translation function f from β to β satisfies*

$$\lim_{x \nearrow \beta} \frac{\beta - f(x)}{\beta - x} = 1. \tag{2}$$

- *Every $\mathbb{R}$-translation function f of bounded variation from β to β satisfies* (2).

Speedability of left-c.e. reals

Definition

A left-c.e. is speedable if there exists its increasing computable approximation $a_0, a_1, \ldots$ and a computable index function f, such that $f(n) \geq n$ for all n, and a constant $\rho < 1$ such that

$$\liminf_{n \to \infty} \frac{\alpha - a_{f(n)}}{\alpha - a_n} < \rho$$

- independent of the choice of ρ and $a_0, a_1, \ldots$;
- Martin-Löf random left-c.e. reals are nonspeedable but not vice versa (Hölzl, Janicki, 2023)

differentiability in a fixed-point

Theorem

Let β be a left-c.e. real.
Then the following statements are equivalent.

- *β is nonspeedable.*
- *Every nondecreasing $\mathbb{R}$-translation function f of bounded variation from β to β satisfies* (2)*.*

The following statements are also equivalent

- *β is Martin-Löf random.*
- *Every $\mathbb{R}$-translation function of bounded variation f from β to β satisfies* (2)*.*

What happens outside of LEFT-CE?

Translation functions and other randomness notions

Conjecture

Let α be a real and β be a Schnorr random real. Then there exists a constant d such that, for every $\mathbb{R}$-translation function from β to α defined on $[0,1]$, it holds that

$$\lim_{n\to\infty} \frac{|\alpha - f(q)|}{|\beta - q|} = d.$$

In particular, it implies that $\alpha \leq^{\text{loc}}_{\text{cL}} \beta$.

What about the converse directions?

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